Examples of Solving Trigonometric equations

Much like how we can potentially solve a quadratic equation by factoring, we can sometimes algebraically solve equations that involve trigonometric functions.

Remember: When we are trying to solve for roots of an equation, we would ideally like to write it as a series of products $(\dots) \times (\dots) \times \dots \times (\dots) = 0$. If we then set each individual product to zero, we can obtain the roots of an equation.

Note: Not all equations have roots. Just like how there is not a solution in the real numbers for the equation $x^2 + 1 = 0$, there are trigonometric equations that do not have a real solution, e.g., $\sin(\theta) = 5$. And just like how sometimes quadratics do not factor nicely, there may not be an easy way to manipulate a random trigonometric expression to solve it exactly.

For the purposes of the Mathematics Skills Test however, if you are asked to solve or simplify something, there will be an algebraic way for you to solve the equation. To give you some idea on how to do this, please see the following examples:

Example 1:

$2\cos^2\theta - 5\sin\theta + 1 = 0$ for $\theta \in [0, 2\pi]$

At first glance, it appears that we do not have an easy way of doing this. However, one thing you should always ask is there another way of writing the expression that might be something you could simplify. In the case of trigonometric equations, one should always consider trigonometric identities.

Recall that $\sin^2\theta + \cos^2\theta = 1$. If we rearrange this equation, we get $\cos^2\theta = 1 - \sin^2\theta$

Now substituting this into our equation we get:

$$2(1 - \sin^2\theta) - 5\sin\theta + 1 = 0 \Leftrightarrow 2 - 2\sin^2\theta - 5\sin\theta + 1 = 0 \Leftrightarrow -2\sin^2\theta - 5\sin\theta + 3 = 0$$

We can divide the entire equation by -1 to get $\Leftrightarrow 2\sin^2\theta + 5\sin\theta - 3 = 0$

If we make the substitution $x = \sin\theta$, this equation can be rewritten as a factorable quadratic.

$\Leftrightarrow 2x^2 + 5x - 3 = 0$

We can factor this as $\Leftrightarrow (2x - 1)(x + 3) = 0$

Substituting back in $x = \sin\theta$ we get $\Leftrightarrow (2\sin\theta - 1)(\sin\theta + 3) = 0$

This means that the solutions to this equation are θs such that either $\sin\theta = \frac{1}{2}$ or $\sin\theta = -3$

Since $-1 \leq \sin\theta \leq 1$ for any $\theta \in \mathbb{R}$ we only need to consider when $\sin\theta = \frac{1}{2}$

The θs between 0 and 2π that satisfy this equation are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

You can use either the special triangles method or the unit circle method to find these angles. Note that if we didn't have a restriction on the domain of θ there would be infinitely many solutions to this equation.

Example 2:

Solve $\sin 2\theta = \cos\theta$ for $\theta \in [0, 2\pi]$

Again, we would like to try to write this as a series of products. In its current form we appear to be stuck. However, we should recall that $\sin 2\theta = 2\sin\theta\cos\theta$

We can therefore write this as $\Leftrightarrow 2\sin\theta\cos\theta = \cos\theta \;\Leftrightarrow 2\sin\theta\cos\theta - \cos\theta = 0$. Since both terms contain $\cos\theta$, we can factor this as $\Leftrightarrow \cos\theta\,(2\sin\theta - 1) = 0$

This means that the solutions to this equation are θs such that either $\sin\theta = \frac{1}{2}$ or $\cos\theta = 0$

In the previous example, we found that for $\sin\theta = \frac{1}{2}$ the θs between 0 and 2π that satisfy this equation are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

So, for $\cos\theta = 0$, you can either use the unit circle method or the properties of the cosine function to know that the θs between 0 and 2π that satisfy this equation are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Thus, the four angles in the domain that satisfy this equation are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$, and $\frac{3\pi}{2}$.

Again, if we did not have a restriction on the domain for this question, there would be infinitely many solutions to this equation.