

From Contour Similarity to Motivic Topologies

Chantal Buteau* and Guerino Mazzola**

*Mathematik Departement, ETH Zürich

**Institut für Informatik, Universität Zürich
and i2musics (Internet Institute for Music Science)

Abstract

This paper is devoted to a qualitative and quantitative study of topological spaces built on premotif collections of musical scores. These motivic topologies are related to similarity concepts in the American music set theory. Through shapes, imitations and gestalts, and similarity relations between any two shapes, we obtain a motivic hierarchy of a score. This model of motivic analysis is completed by proposing different tools, such as weighted shapes and motivic evolution trees (MET), for visualizing these non-intuitive topologies. The concept MET, which is related to a systematic variation of the similarity parameter, is of a highly cognitive flavor.

In the light of our approach, the still debated question concerning the length of the main theme in Bach's *Kunst der Fuge* is addressed. As a result, we can state that the extended 12-tone theme is essentially equivalent to the shorter 8-tone version when viewed from its motivic substance.

The rigorous language of mathematics has become a powerful tool in musicology. It typically helps understanding complex mechanisms by making abstraction from secondary qualities of the studied objects. In this spirit, Allen Forte (1973) used set theory to model the structure of atonal music. His theory was later adapted and extended to study motivic structures in music by Robert Morris (1987), John Rahn (1980), David Lewin (1980), and others, and also more computing-oriented (e.g. Selfridge-Field (1997)), see also the works of Dominik Hörnel and Thomas Ragg (1996), Emiliós Cambouropoulos (1997), and others (e.g. Hewlett W.B. and E. Selfridge-Field (1998)).

In this musical set theory, contours are introduced as an abstraction to describe motivic phenomena. While contour equivalence is a straightforward relation, similarity between contours is a concept that is still difficult to manage. For example, in Ian Quinn's approach (1997) using fuzzy logic, a contour within a given collection of highly similar contours may fail to be "viable" in this collection, a condition which should be automatic for any reasonable definition of "viability".

Where does this problem come from? The set-theoretic approaches used in contour theory succeed in grouping equivalent contours and give numerical values for similarity between 'some' pairs of contours. But similarity values do not restructure a score-related collection of contours nor regroup or organize them in a hierarchy.

In order to 'harmonize' the concepts of equivalence and similarity, and to organize a determined collection of contours in a global structure, we propose a topological approach, i.e., we introduce the structure of a mathematical topological space on the set of a score's contours. This topology entails a contour hierarchy for each choice of a number of 'system' parameters. It is an important idea of this topological approach that connections between contours of different cardinalities are possible as an essential extension of 'usual' theories on similarity of '*n*-contours'.

A subtle and consistent vocabulary for motivic analysis is established. We obtain qualitative and quantitative results by use of facts from general topology (e.g. Kelley J. (1955)). This approach is meant as a starting point for further research on grouping and on the thematic content of a score. Because the yoga of our approach is to construct motivic semantics and not to presuppose it, a motivic analysis of 'immanent character' is achieved (See Section 1). A software implementation of our method is available in the module MeloRubette[®] of the software RUBATO[®] (<http://www.ifi.unizh.ch/groups/mml/musicmedia/rubato/rubato.html>; Mazzola G. and O. Zahorka (1994)). This means that time-constraints and complexity of our method have been already dealt with. Moreover, results from applications of our methodology (through RUBATO[®]) to different scores, such as Schumann's *Träumerei* (Mazzola G., O. Zahorka. and J. Stange-Elbe (1992)), Bach's *Kunst der Fuge* (Stange-Elbe J. and G. Mazzola (1998)), and Webern's *Variation für Klavier op 27/2* (Beran J. and G. Mazzola (2000)), support the methodology's validity.

To illustrate this topological method we have analyzed the main theme of Johann Sebastian Bach's *Kunst der Fuge*. This subject is still of interest because there is a classical (and unresolved) debate on the question whether the theme ends after eight notes or whether it extends to twelve notes (see Figure 1). To prevent misunderstandings we emphasize on the fact that we inquire the theme as an isolated entity. We do not look at its role within the whole composition. We rather are interested in its power as an autonomous germ, that is its interior structure, its substance.



Figure 1. Main theme of Bach's *Kunst der Fuge*, 8-tone and 12-tone version.

We therefore took a closer view at its motivic 'anatomy' that is generated in our topological framework. We compared the structures of the motivic topology of the 8-tone theme with the corresponding structures of the 12-tone theme (no comparison with the rest of the score). The result can be stated as follows:

The significant contours of the 8-tone theme are part of the significant contours of the 12-tone theme, but the last four notes do not generate a proper extension to the set of significant contours. However, the last four notes are all related to the significant contours of the 12-tone theme. In other words, the extension to twelve tones is 'substantial', but it is not a proper extension.

Observe that our concept of a 'significant contour' is a highly sophisticated mathematical concept. This means that the preceding result may confirm musicological intuition but it is a completely rational fact, which significantly transcends prescientific knowledge.

In section 1, we have a look at the musicological situation to be modeled. In section 2, the first part of the mathematical system is presented in the musical set theory approach, but with a slightly different language. After a short introduction to general topology in section 3, the construction of motivic topologies is presented in section 4, including the calculus of motivic weights. In section 5, we introduce the main theme of *Kunst der Fuge* within the context of this theory. In section 6, we develop a more in-depth analysis by use of motivic evolution trees (MET), a concept which is related to the systematic variation of the similarity parameter of this analysis. The MET theory is finally applied to the Bach's main theme.

This paper has also been written in order to present our approach as related to musical set theory. Therefore, the reader should be aware that this exposition is a specialized version of a more general topological theory of musical motives (Buteau C. (1998); Mazzola G. and O. Zahorka (To appear); Mazzola G. (To appear)), which includes Morris' contour concepts.

1 The Problem of Modeling the Motivic Structure of a Score

In traditional musicology, the analysis of the motivic organization of a composition is an important task, for instance in understanding the motivic and thematic processes ("motivisch-thematische Arbeit") in Ludwig van Beethoven's or Robert Schumann's work. Following Rudolph Reti (1951) and Michael Kopfermann (Metzger H.-K. and R. Riehn (1982): See the final remark in Kopfermann's translation of Reti's work on Schumann's *Kinderszenen*), the main goal of such a theory would be to recognize the semantics of motivic units, i.e., to find out which sequences of tones are the germs and motors in the evolution of the motivic and thematic content. More

precisely, Reti's approach is that of an autonomous analysis: We do not impose the germinal motives from outside, but have to construct the germs from a thorough analysis of all possible motif structures and relations within a given composition.

However, Reti's discourse is everything but formal. He is more concerned with the possible construction of semantics than with precise definitions of motivic attributes, such as contour, similarity or equivalence. In his work ((1951): See his footnote on page 12), he even refuses to establish strict definitions:

In general, the author does not believe in the possibility or even desirability of enforcing strict musical definitions. Musical phenomena come to existence in the constant fluence and motion of compositional creation. Therefore any description of theme must finally prove but approximations.

Nonetheless, Reti's discourse makes use of semi-technical terms, such as "shape", "imitation", "variation", "transformation", and shape "identity". Along these fuzzy concepts, one can learn that the problem is less that of strict definition, rather is it the expression of a deeper phenomenon: Reti's concept of identity of motif shapes includes relations—such as variation, which is a kind of similarity—which are not necessarily transitive. But any reasonable concept of "identity" should be transitive (in fact should be an equivalence relation: reflexive, symmetric, and transitive). At the same time, Reti's usage of "identity" includes limits which are imposed by variations and which one could relate to metrical distance.

So basically, Reti's problem is the need for a concept that is 'between' plain identification and fuzzy association. In terms of modern music semiotics, Reti is searching for a paradigmatic theme (e.g. Nattiez J.-J. (1975)), i.e., a well-defined associative field of melodies built around any fixed melody. Our approach deals exactly with this objective: To define precisely what is a shape, and to construct associative fields of melodies that are in state of eventually yielding the required semantics of motivic germs.

2 Equivalence and Similarity of Motives

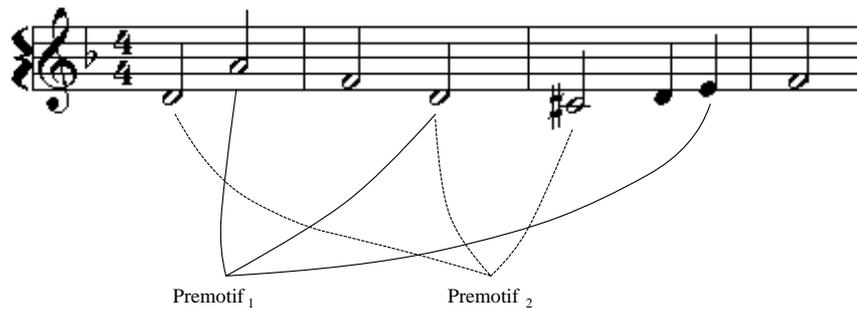
The main objects of our study are short melodic units which we call pre-motives. Intuitively, a pre-motif is a set¹ of tones in which only one tone occurs at a given onset time, and in which tones are not necessarily consecutive in the given composition. This last property is in fact essential for any serious motivic analysis since the tones of a potentially significant pre-motif might be separated from each other by secondary or ornamental tones. For example (as Rudolph Reti (1951) remarks), the beginning of Schumann's first piece of *Kinderszenen* shows accents on notes at the end of bars 1,2,3, a sequence that is defined by non-consecutive peak notes. More generally in the spirit of Schenker analysis, motivic structures share hierarchical groupings which automatically enforce 'higher-order' pre-motives of non-consecutive notes.

Pre-motives are not necessarily germs of a composition, but only a priori candidates for carrying such a motivic meaning. We introduce the prefix '*pre*' in order to make clear the difference between the formal structure of the mathematical theory and musically significant motivic germs, the 'real' motives.

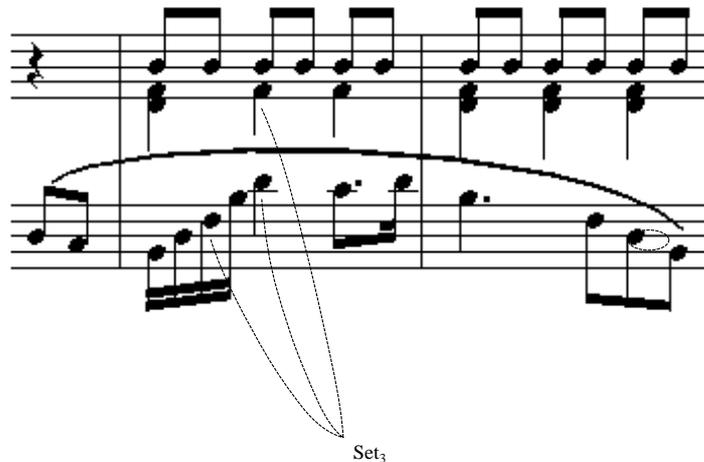
¹ Observe that we always use the set concept in its strictly mathematical sense, i.e., as a non-ordered collection of objects.

The formal definition of a premotif runs as follows (see Figure 2):

A premotif M is a non-empty finite set $M = \{m_1, \dots, m_n\}$ of n tones where each tone m_i is a couple $m_i = (o_i, p_i)$ of real numbers² representing the tone's onset o_i and pitch p_i . We impose the condition on M that any two different tones $m_i \bullet m_j$ have different onsets $o_i \bullet o_j$. A premotif M^* is called a subpremotif of a premotif M if it is a subset of M . A premotif of cardinality n is also called a n -premotif. The set of all possible n -premotives is denoted by MOT_n ; the set of all premotives MOT is the disjoint union $MOT = \coprod_n MOT_n$. The space of tones is denoted by OP .



These two sets of tones form respectively premotives. However, the set containing the three tones



is not a premotif.

Figure 2. Example of premotives and non-premotives in a score. Recall that these 'mathematical' premotives are 'only' candidates for being the musically meaningful motives of a score.

Quite generally, if one compares premotives, one concentrates on certain relevant "shape" properties. Here, we want to consider the properties as introduced in contour theory (e.g. Morris R. (1987)). More precisely, we refer to the COM-matrices that describe the diastematic movement between the premotif's tones. Recall that for a n -premotif M , $COM(M)$ is the $n \times n$ -matrix $(b_{i,j})$ whose coefficient $b_{i,j}$ at row i and column j is 1 if $p_j > p_i$, -1 if $p_j < p_i$, and 0 if $p_j = p_i$.

² Elements of the set \mathbf{R} of real numbers.

Note that the diagonal is always identically zero, and that the coefficients below the diagonal are just the negative values of the corresponding coefficients above the diagonal, i.e., $b_{j,i} = -b_{i,j}$. This redundancy is the reason why the CSIM calculation in contour theory uses only the coefficients above the diagonal. This motivates our concept of shape type associated with the COM concept (See Figure 3).

To this end, we consider all possible sequences of $n(n-1)/2$ elements³ with values $-1, 0,$ and 1 . This set of sequences is denoted \bullet_n . A n -shape $(a_1, a_2, \dots, a_{n(n-1)/2})$ represents the upper triangle of a COM matrix. We now define the Com shape type as being the family of mappings

$$Com_n : MOT_n \rightarrow \Gamma_n = \{-1, 0, +1\}^{n(n-1)/2} : M \mapsto Com_n(M)$$

sending a n -premotif M to the shape

$$Com_n(M) = (b_{1,2}, b_{1,3}, \dots, b_{1,n}, b_{2,3}, \dots, b_{n-1,n})$$

of the upper triangular values of $COM(M)$. The disjoint union $\bullet = \coprod_n \Gamma_n$ is called the space of shapes (with respect to shape type Com)⁴.

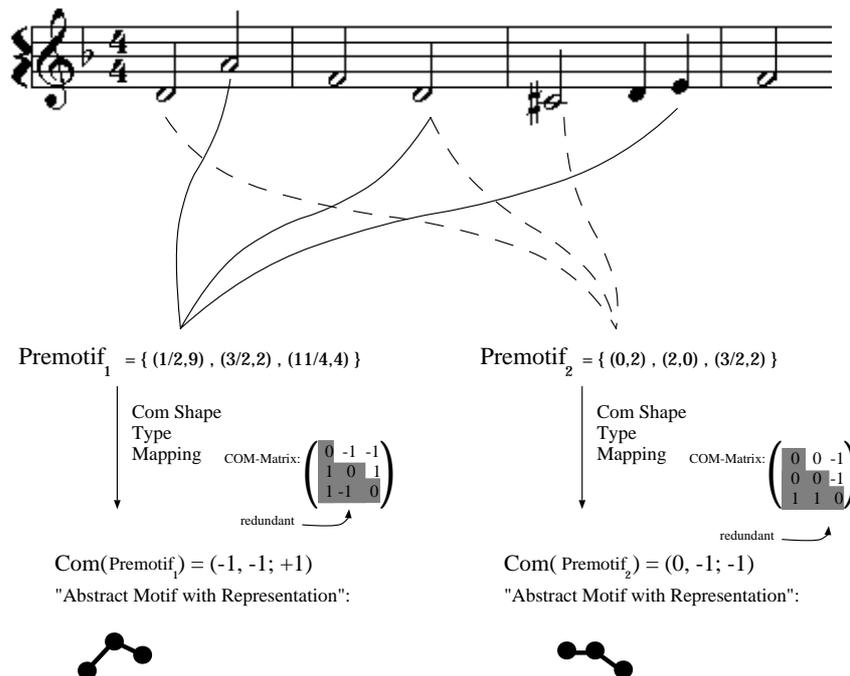


Figure 3. The shape construction for two premotives which are taken from the theme of *Kunst der Fuge*: First construct their COM-matrices, retain only the upper triangular values, then lay them out as sequences.

³ Number of elements of the triangular part above the diagonal of a matrix COM for a n -premotif.

⁴ The *Com* shape type may then be identified with the disjoint union $Com = \coprod_n Com_n : MOT \bullet \bullet$.

We can now proceed to discuss 'transformationally equivalent' shapes (c-space segment classes (e.g. Morris R. (1987))). The general situation is as follows. We consider a group of musical transformations, such as inversion or retrograde, operating on shapes. Such a group allows to assemble all premotives whose shapes can be derived from each other by any of these transformations (see Figure 4). Such an ensemble is called a gestalt.

In this context, we shall focus on very special groups: subgroups CP of the affine counterpoint group ACP. This group is the subgroup of transformations of the space OP of tones, which is generated by all transpositions, time shifts, pitch inversions, and retrogrades. For any n-premotif $M=\{m_1,\dots,m_n\}$, a transformation p in CP defines a transformed premotif, $p \cdot M =\{p(m_1),\dots,p(m_n)\}$.

The affine counterpoint group also operates in a natural way on the shape space \bullet_n . More precisely, for any shape $Com_n(M)$ of a premotif M, we set $p \cdot Com_n(M) = Com_n(p \cdot M)$.

With these definitions, the gestalt Ges(M) of a n-premotif M is defined as the inverse image $Ges(M) = Com_n^{-1} (CP \cdot Com_n(M))$ of the orbit $CP \cdot Com_n(M)$ of M. Since a n-premotif N is in the gestalt of n-premotif M if and only if there is a counterpoint transformation $p \in CP$ such that $p \cdot Com_n(N) = Com_n(M)$, the set of all premotives is partitioned into mutually disjoint gestalts.

So the total set MOT of premotives is given an equivalence relation by the above gestalt concept: Two premotives are gestalt-equivalent if their gestalts coincide. Further, the space \bullet of shapes is also given an equivalence relation by the orbits of the operation of the group CP. Observe that the gestalt relation and the orbit relation are compatible under the Com mapping.

Along this paper, we shall focus on two groups CP, the first being the full group ACP, and the second being the group TR of all translations (transpositions, time shifts, and combinations thereof).

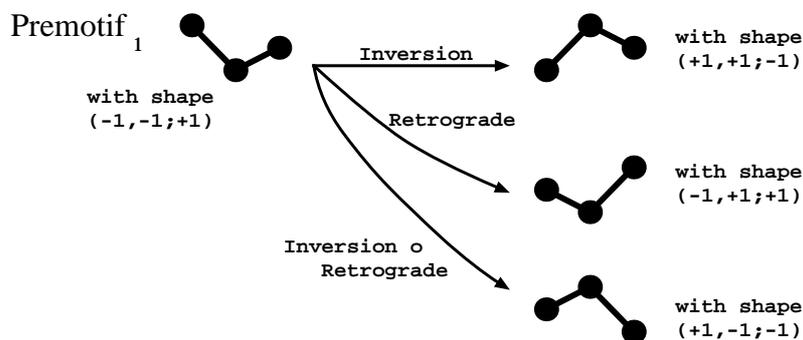


Figure 4. Transformations of *Premotif₁* (see Figure 2) together with the contrapuntal transformations of the associated shapes. We observe that the shape of a transformation of *Premotif₁* corresponds exactly to the transformation of *Premotif₁*'s shape. By collecting all 3-premotives having one of these shapes we obtain the so-called gestalt of *Premotif₁* (or of any of

these premotives).

So now, shapes and premotives, respectively, are linked by equivalence relations. In the next step, we would like to be more 'permissive' by linking premotives that have "highly similar" shapes. So premotif similarity is induced by shape similarity that can be conceived in various ways, for example by use of CSIM or C⁺SIM values (e.g. Morris R. (1987)). We shall be using a variant of this approach (see Figure 5). For two n -shapes $S = (s_1, \dots, s_{n(n-1)/2})$, $T = (t_1, \dots, t_{n(n-1)/2})$, our value is (up to the constant factor $\sqrt{2}/n$) the Euclidean distance function:

$$EUCSIM(S, T) = \sqrt{\frac{\sum_{i=1}^{n(n-1)/2} (s_i - t_i)^2}{n^2/2}}$$

where we take into consideration the mean square difference of shape coordinates. This distance function has the properties that define a *metric*⁵ on the space \bullet_n of n -shapes.

We then define the distance $d_{Com,n}$ between two n -premotives M and N as being the *EUCSIM* value of their respective shapes:

$$d_{Com,n}(M, N) = EUCSIM(Com_n(M), Com_n(N)).$$

This distance function $d_{Com,n}$ on the set MOT_n satisfies all properties of a metric except that $d_{Com,n}(M, N) = 0$ does not necessarily imply $M = N$, whereas we still always have $d_{Com,n}(M, M) = 0$. This is clear since two different premotives may have the same shape, and therefore vanishing distance. Such a weaker distance function defines a pseudo-metric on MOT_n .

Finally, we can define the gestalt distance $gd_{Com,n}(M, N)$ between two n -premotives M and N as the minimum of all distances $d_{Com,n}(M_1, N_1)$ between all possible pairs of premotives $M_1 \in Ges(M)$ and $N_1 \in Ges(N)$.

Distance between premotives:

$$\begin{aligned} d_{Com,3}(premotif_1, premotif_2) &= EUCSIM((-1, -1; 1), (0, -1; -1)) \\ &= \sqrt{\frac{(-1-0)^2 + (-1-(-1))^2 + (1-(-1))^2}{3}} = \frac{\sqrt{5}}{3} \end{aligned}$$

Gestalt distance between premotives:

$$\begin{aligned} gd_{Com,3}(premotif_1, premotif_2) &= \text{Min}\{EUCSIM((-1, -1; +1), (0, -1; -1)), EUCSIM((+1, +1; -1), (0, -1; -1)), \\ &\quad EUCSIM((-1, +1; +1), (0, -1; -1)), EUCSIM((+1, -1; -1), (0, -1; -1))\} \\ &= \text{Min}\left\{\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}, 1, \frac{1}{3}\right\} = \frac{1}{3} \end{aligned}$$

Figure 5. The similarity value between *Premotif₁* and *Premotif₂* (see Figure 2) is given through a

⁵ I.e. $EUCSIM(S, T) = 0$ if and only if two n -shapes S, T are equal, $EUCSIM(S, T) = EUCSIM(T, S)$ for all couples S, T (symmetry), and $EUCSIM(S, T) + EUCSIM(T, W) \geq EUCSIM(S, W)$ for all triples S, T, W of n -shapes (triangular inequality).

variant of their CSIM value. As a similarity value on their gestalt level we transform these two premotives until we reach their most similar shapes and take this 'best' similarity value. It can be shown (Buteau C. (1998)) that it is sufficient to fix one shape and to make the other one vary in order to determine the minimal distance.

Observe that this distance function coincides with the distance function between c-segment classes (e.g. Morris R. (1987)) since translations operate trivially on shapes (!), and we may therefore look at inversions, retrogrades, and retrograde inversions of premotives within their gestalts. We should also observe that the gestalt distance between two premotives is 0 if and only if they share the same gestalt, i.e., if they are transformationally equivalent in the sense of music set theory.

Again, the distance $gd_{Com,n}$ on MOT_n satisfies the pseudo-metric properties (Buteau C. (1998)).

The gestalt distance among n -premotives is a natural candidate for defining similarity. Given a positive distance number s , we may call two n -premotives M, N s -similar, if their gestalt distance $d_{Com,n}(M,N)$ is smaller than s . The similarity relation is reflexive, symmetric, but however not transitive. Because of the failure of this last property, we cannot regroup premotives, as we did with the gestalt construction: s -similarity is not an equivalence relation. By use of this relation, we cannot build similarity classes without getting into inconsistent situations. For example, a premotif M may be s -similar to a premotif N and to a premotif O without N being necessarily s -similar to O . This is a precise restatement of Reti's problem of the identification of premotives, see section 1: Similarity is not transitive, and so is Reti's concept of identity.

So far, we have restated the known approaches to similarity in terms of pseudo-metrics. But how can we evaluate the similarity between 'any' two premotives (possibly with different cardinalities) or 'any' two shapes? How could we manage to have premotives, their relatives with respect to counterpoint imitation, and their *highly* similar associates all in the same global structure without ending up in Reti's cul de sac? A solution of these problems lies in the use of topologies.

3 What is a Mathematical Topology?

In fact, the philosophy of the classical and basic mathematical theory of topologies is to yield an axiomatic description of general phenomena of "similarity" and to provide us with concepts that help understanding such structures. Topology is one of the richest mathematical theories and has helped understanding all kinds of deformation phenomena, such as counting "holes of doughnuts", or understanding the "missing orientation on the Moebius strip".

Topology is the general theory of spaces that are not necessarily provided with the geometric evidence of the common ("Euclidean") space of physical reality. Whenever there is a metrical (or even a pseudo-metrical) distance structure on a space of objects, there is topology (we shall give its precise description in a moment), but not conversely: Topology is less intuitive than metrical structure. And this is what we shall need in the sequel.

The rigorous definition of a topology runs as follows (For a classical reference, see Kelley J. (1955)). Intuitive presentations may be found in any undergraduate textbook on calculus). Given a set X (the "space" of our topology), a topology for X is a collection \mathcal{T} of subsets of X such

that these subsets 'behave well' in the sense that the intersection of any two such subsets or the union of any collection of such subsets is also an element of \mathcal{T} . More precisely:

A topology for a set X is a collection \mathcal{T} of subsets of X such that

1. $X \in \mathcal{T}$,
2. $A_1 \cap A_2 \in \mathcal{T}$ whenever $A_1, A_2 \in \mathcal{T}$,
3. $\cup A_i \in \mathcal{T}$ for all families (A_i) of elements $A_i \in \mathcal{T}$.

The set X is called the space of \mathcal{T} , (X, \mathcal{T}) a topological space, and members $A \in \mathcal{T}$ are called the open sets of the topology \mathcal{T} . For any element $x \in X$, a set $U \subset X$ such that there is $V \in \mathcal{T}$ with $x \in V \subset U$, is called a neighborhood of x (in the topology \mathcal{T}). A subset B of \mathcal{T} is a base for \mathcal{T} if every open set is a union of some elements of B .

A classical example of a topology that is defined by a metrical distance function is provided on the real plane (see Figure 6). We take the usual Euclidean distance $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ of points $x = (x_1, x_2), y = (y_1, y_2)$. This topology is defined by use of disks. For a positive radius r , the disk of radius r around a point x is the set $D_r(x) = \{y, d(x, y) < r\}$. Then, by definition, a set O is open, if for each of its points x , there is a 'small' disk $D_r(x)$ of positive radius r around x which is entirely contained in O . In other words, open sets are just unions of disks. The set of all disks is a base of this topology. It is a mathematical fact (e.g. Loomis L.H. and S. Sternberg (1968)) that we obtain the same topology if we replace the distance $d(x, y)$ by any other distance for a metric, i.e. the metric cannot be recovered from the associated topology. The reader should recall this nontrivial fact when we introduce motivic topology that refers to determined distance functions.

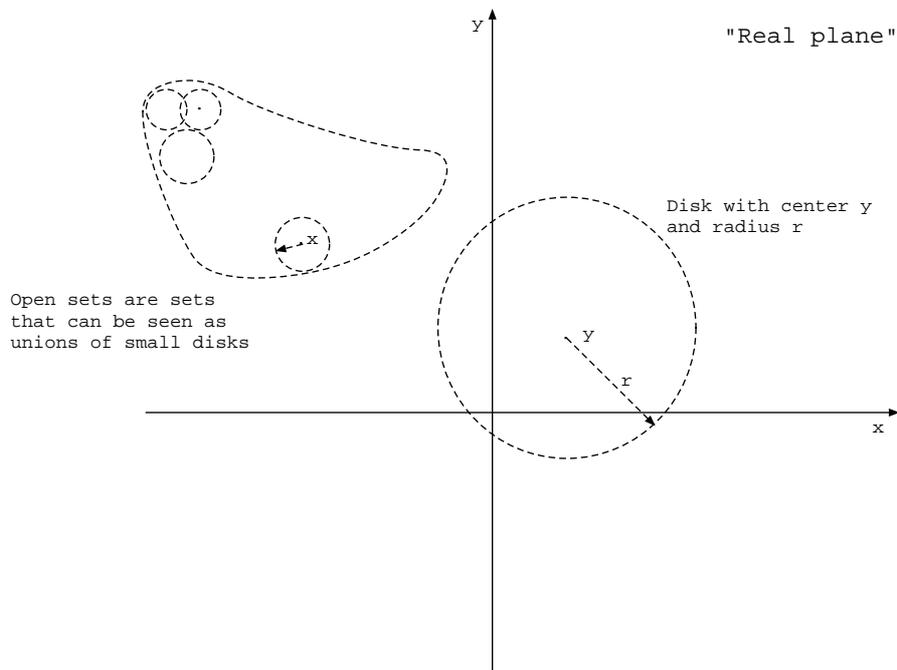


Figure 6. The usual topology on the real plane \mathbf{R}^2 is based upon disks that are defined by the use

of the Euclidean distance. The essential concept is 'closeness' of points. In this intuitive topological space 'close points' are easily viewed as being in a same disk for a 'small' radius r . Such a trivial geometric representation of closeness is however not possible in the general case of topologies, such as for our motivic topologies.

The intuition we get from Figure 6 says that a base is a collection of 'small open sets' capable of 'filling out' every open set of our topology. And a point's neighborhood conveys the idea of 'what is around the point'. But the intuition from common spaces is quite misleading: In 'pathological' spaces, neighbors can be so 'close' to each other that one cannot 'separate' them. This will be the case in our motivic topologies: they are far from being intuitive. For this reason, we will have to make an extra effort for their visualization.

4 The Motivic Topology of a Score

Let us recapitulate the status of the situation generated by gestalt distances between premotives. For each premotif cardinality n , the space MOT_n of all n -premotives is provided with the pseudo-metric of the gestalt distance $gd_{Com,n}$. So, according to the above discussion of topologies, this space also carries the topology which is associated with $gd_{Com,n}$. However, we are not only interested in giving each 'layer' MOT_n of premotives a separate topology, we want to compare premotives of different cardinalities: We would like to say that a premotif M is "similar" to another premotif N if N , after removing some 'bad tones', is s -similar to M for some gestalt distance s . This will extend similarity beyond the limits of a fixed premotif cardinality.

The formal definition of the motivic topology runs as follows. Like with the topology of the real plane (see Figure 7), we first define the "disk neighborhoods" around each premotif M . For a positive radius r , the r -neighborhood of M is the following set:

$$V_r(M) = \{N \in MOT, \text{ there exists } N^* \subseteq N \text{ with } gd_{Com}(M, N^*) < r\}.$$

Then the open sets of the motivic topology \mathcal{T} on MOT are the unions of any collection of such neighborhoods (the radii may vary).

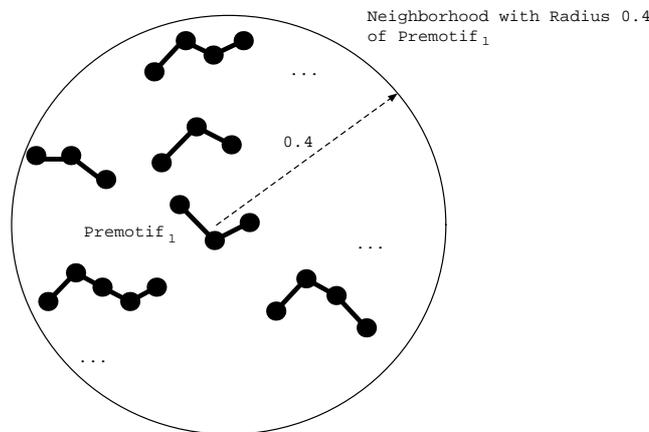


Figure 7. The disk-like 0.4-neighborhood of $Premotif_1$ from Figure 2. This neighborhood contains all 3-premotives that are less than 0.4 close to $Premotif_1$, and also all premotives with

more than 3 tones and that contain a subpremotif which is less than 0.4 close to *Premotifi*.

So the r -neighborhood of a premotif contains its counterpoint imitations from the group CP as well as its r -similar premotives, and all the premotives with higher cardinality in which the premotif is "included up to r -similarity". Moreover, such a neighborhood contains the gestalt of each of its premotives; it is "stable under gestalt formation".

It is however not obvious that the above construction does define a topology. Although the topology axioms 1 and 3 (see Section 3) are evidently satisfied, the second one is not evident. We essentially have to show that the intersection

$$V_{r_1}(M_1) \bullet V_{r_2}(M_2)$$

of any two r -neighborhoods is an open set. This is true because we have chosen the *Com*-shape type. If instead we had chosen Morris' contour vector INT_1 (e.g. Morris R. (1987)) (which is the vector $(a_{1,2}, a_{2,3}, \dots, a_{n-1,n})$ in the COM matrix), the above definition would not yield a topology. The reason for this failure is that the so-called *inheritance property* fails, whereas it is satisfied for the *Com*-shape type. Intuitively speaking, this property requires that the similarity of premotives implies the similarity of their subpremotives. In fact, the contour vector INT_1 disregards subpremotif information with respect to the vector's total information: Knowing a 3-tone premotif's vector, e.g. $(1,-1)$, does not carry any information about the vector of the subpremotif consisting of the first and third tone.

So far we have been working out the topology \mathcal{T} on MOT , the collection of all possible premotives. But how do we obtain the topology of a specific score? Recall that in Reti's approach, we should exclusively deal with premotives taken from the score: the analysis is meant to be an intrinsic one. For this reason, we should determine which premotives of the score are to be selected for the present analysis. We denote this subcollection of MOT by MOT^∇ . In general, many different selection criteria can be imposed, be it for musicological reasons, be it for reasons of calculation time, or for cognitive rationales. In the context of this paper, we shall take all the score premotives with cardinality between (inclusive) 2 and a maximal cardinality. Then the motivic topology of a score selection is simply the motivic topology \mathcal{T} "restricted" to our premotif selection from the score:

The motivic topology \mathcal{T}^∇ of the score selection MOT^∇ is the relative topology of \mathcal{T} with respect to the subcollection $MOT^\nabla \subset MOT$. By definition, the open sets of the relative topology are the intersections $O \bullet MOT^\nabla$ of open sets $O \in \mathcal{T}$ with the selection MOT^∇ .

We should observe that the motivic topology is much less intuitive than the topology that we have encountered on the real plane. In fact, for any two points of the real plane, there are always disk neighborhoods of these points that have no common points. For the motivic topology, however, this is no longer possible in general: For example, if we take a proper subpremotif M of N , the premotif N is in every neighborhood of M , whereas M is in no r -neighborhood of N . Intuitively speaking, N is "infinitely near" to M , but not conversely! In mathematical life, this type of counter-intuitive topologies is not the exception: the entire world of algebraic geometry is characterized by such topological phenomena.

For this reason, we have to elaborate a more intuitive description of these topologies in order to create a valuable visualization. In particular, we have to take into account the intrinsic neighborhood asymmetry between premotives, as discussed above. The decisive feature of this asymmetry is that premotives are inserted in a hierarchical structure, and this is what we have to work out now.

So the configuration of neighborhoods of an individual premotif in MOT^∇ strongly depends on the individual premotif. We want to provide a more explicit, viz., quantitative expression of this information. This will eventually lead to understand the significance of a premotif in the selected collection. Our approach uses a weight function such that—in a first approximation—the heaviest premotives are the most important ones.

The premotif's weight is a real number, basically obtained from the magnitude of its neighborhood (its 'presence'), and from the frequency of its appearance in other premotives' neighborhoods (its 'content'). Since neighborhoods are built from r -neighborhoods, we want to fix a neighborhood radius r to define the premotif weights (recall from the end of section 2 that fixing a neighborhood radius imposes a limit for premotif similarity).

The formal definition of a premotif's weight is the following: Let $M \in MOT_m^\nabla$, $N \in MOT_m^\nabla$. We set

$$pr_r(M, N) = \frac{1}{2^{n-m}} \cdot \text{Card}\{N^* \subseteq N \text{ with } n^* = m, \text{ and } gd_{Com}(M, N^*) < r\} \text{ and}$$

$$ctr(M, N) = \frac{1}{2^{m-n}} \cdot \text{Card}\{N^* \subseteq M \text{ with } n^* = n, \text{ and } gd_{Com}(N, N^*) < r\}.$$

The presence and the content of the premotif M at neighborhood radius r are

$$presence(M, r) = \sum_{N \in MOT^\nabla} pr_r(M, N) \text{ and } content(M, r) = \sum_{N \in MOT^\nabla} ctr(M, N).$$

Bringing these two definitions together we obtain the weight of a premotif M for a neighborhood radius r :

$$Weight(M, r) = presence(M, r) \cdot content(M, r).$$

Although there is no proof of any formal definition, we should make our *Weight* function plausible. To begin with, the presence of a premotif M is a measure for the number of premotives that are r -similar to M and appear in larger premotives of our composition. Second, the content of M measures the number of smaller premotives that are r -similar to a subpremotif of M . Thirdly, the weight of a premotif should be proportional to its presence if its content is fixed (e.g. double weight for double presence and fixed content); this leads to the product formula for the motivic weight.

We have now completed our topological model on motivic analysis of a score, together with its quantification by premotif weights. We recall that in order to present our approach in a

'simple' form we fixed many parameters, such as the counterpoint group⁶, or the function *Weight*⁷, the latter having more degrees of freedom in the general theory. In the next section we present a detailed example of a motivic topology for a short melody: the main theme in Bach's *Kunst der Fuge*.

5 Introducing the Motivic Topology of the Main Theme in Bach's *Kunst der Fuge*

This example was programmed in Mathematica[®]. The set MOT^∇ of premotives to be analyzed contains all the possible 2- to 8-tone premotives in the 8-tone main theme of *Kunst der Fuge* (see Figure 1), i.e., all the possible temporal sequences of two to eight (not necessarily adjacent) tones. As previously mentioned, premotives are *not automatically* 'semantic' germs of the composition, but only a priori candidates for carrying such a meaning.

Recall that in this paper, premotives are viewed with respect to the *Com*-shape type (derived from COM-matrices) and to orbits of either the group $CP = TR$ of time and pitch translations, or the full counterpoint group $CP = ACP$ (see Table 1 for details).

**Number of Premotives and Gestalts of
Kunst der Fuge's 8 Tone Main Theme**

Motif Cardinality	# Premotives	# Gestalts (<i>TR</i>)	# Gestalts (<i>ACP</i>)
2	28	3	2
3	56	13	5
4	70	30	18
5	56	38	34
6	28	25	25
7	8	8	8
8	1	1	1

Table 1. Number of premotives and gestalts for each premotif cardinality and for both, translation and counterpoint groups in Bach's *Kunst der Fuge*'s 8 tone main theme.

Also recall that r -similarity between n -premotives is calculated by means of the *EUCSIM* distance function. In the case of the translation group TR , the minimal gestalt distance gd_{com} between two premotives of different gestalts is 0.202, and the maximal one is 1.732. For the counterpoint group ACP , the minimum is again 0.202, and the maximum is 1.22.

In order to visualize the motivic topology and hierarchy of the 8-tone main theme of *Kunst*

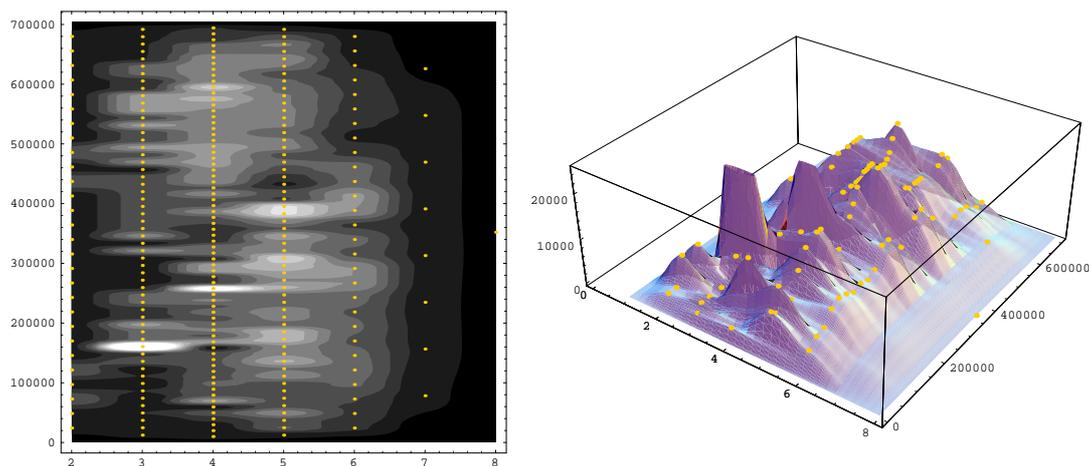
⁶ Instead, we could have chosen other groups of transformations; for example groups that contain no retrograde motion.

⁷ Other functions quantifying the topology in a 'reasonable' way can be used. For example they could be motivated by arguments from cognitive science.

der Fuge, we first order the premotives according to their cardinality, and for fixed cardinality, we order them lexicographically with respect to the temporal order of their tones. We then arrange the premotives on a grid where the first coordinate indicates the premotives' cardinalities, whereas the second coordinate indicates their lexicographic order (see Figure 8). Of course, this premotif order is a strictly graphical device and does not influence the topology nor the premotif weights. Finally, the weight function, which is only defined on the discrete set of premotives, is interpolated to a continuous surface in order to make evident the overall weight distribution.

According to the above limit distances, weight functions are evaluated for neighborhood radii varying from 0.2 to 1.65 for the translation group *TR*, and from 0.2 to 1.25 for the full counterpoint group *ACP*, and taking in both cases a step width of 0.05. This means that we calculate the weight functions for radii = 0.2, 0.25, 0.3,... etc. This step width is an empirical choice that we had to take for reasons of calculation time. The ideal procedure would be to take just all the successive distance values where a change in the weight configuration happens.

For example, when choosing the neighborhood radius to be 0.85 and the translation group, we obtain the weight interpolation surface as shown in Figure 8.



At Neighborhood Radius = 0.85:



Figure 8. This figure shows the weight function for the set of premotives in the 8-tone main theme of *Kunst der Fuge*. The weights are calculated for neighborhood radius = 0.85 and for the translation group. In the left graphic, the premotives are represented as a set of small white points. They are arranged in a grid, with horizontal axis for cardinality, and vertical axis for the lexicographic order (the lexicographic scale numbers are not relevant). The surface gray levels indicate relative weights: High weights are shown in light gray, low weights appear in dark gray. The graph on the right is the same weight function but seen as landscape of mountains from a more general perspective; the peaks show premotives with high weights. Below the graphs, we

show five premotives and their 0.85-similarity relations: in the upper row, all three 3-premotives are similar, whereas the two 4-premotives below are not similar.

6 Motivic Evolution Trees of the Main Theme in Bach's *Kunst der Fuge*

At this point, we should make a methodological remark concerning the variety of available or reasonable radii for the described weight analysis. The selection of a radius (much as the selection of a transformation group and the shape type!) is susceptible of cognitive relevance. In other words, the selection of a radius could mean that we want to express "cognitive perception of similarity" among premotives. For such an empirically founded situation, the selection of a radius would be justified.

But our goal is not to apply cognitive science and its results, rather are we involved in an overall, integrated image of what happens, a conceptual framework that could be used to develop cognitive models. Our goal is therefore to describe the entire 'spectrum' of weight landscapes as it is produced when one moves along the axis of similarity radii.

In what follows, we want to describe this 'weight spectrum' which we call the *Motivic Evolution Tree* (MET). To make our ideas concrete, we explain this concept on the example of the 8-tone theme of *Kunst der Fuge*, and we stick to the group *TR* of translations.

The basic material for our construction is the entire collection of the weight functions for all selected radii $r = 0.2, 0.25, \dots 1.65$. For every weight function in this list, we pick all the premotif gestalts with the highest weight and all the premotif gestalts with the second highest weight (recall that the weight does not change among the members of a fixed gestalt). Then, we arrange these top gestalts in the 'coordinate system' of the radii (vertical axis) and the premotif cardinalities (horizontal axis), see Figure 9.

The black gestalt representatives are associated with top weights, the gray ones pertain to second highest weights. In our figure, each premotif carries two flags. The upper flag (e.g. 4-26) is its cardinality (4) and the lexicographic number of its gestalt (26). The lower flag is the number of the representatives of the premotif's gestalt within the 8-tone main theme.

By default, a heaviest (resp. second heaviest) gestalt keeps this role with growing radii until a new heaviest (resp. second heaviest) gestalt appears. For example, gestalt 3-13 is heaviest for the minimal radius, and remains in this role until gestalt 4-26 takes the leading position at radius 0.4. At radius 0.5, gestalt 3-13 comes back to the leading position, that is why it is shown in a dashed box. If more than one gestalt appears as top weighted candidate, we draw a box around this gestalt group.

Finally, focusing again on the topological perspective, we underline the inclusion relations between gestalt representatives. If we consider two gestalts for consecutive cardinalities (2 and 3, say) in our graphical scheme, a line connects them if in each of these gestalts, there is a premotif such that the smaller premotif is contained in the larger one. For example, gestalt 2-1 is connected to gestalt 3-8, which is easily recognized from the premotives that are shown in the scheme. Since this *gestalt inclusion relation* is transitive, we may deduce other inclusions by concatenation of connecting lines.

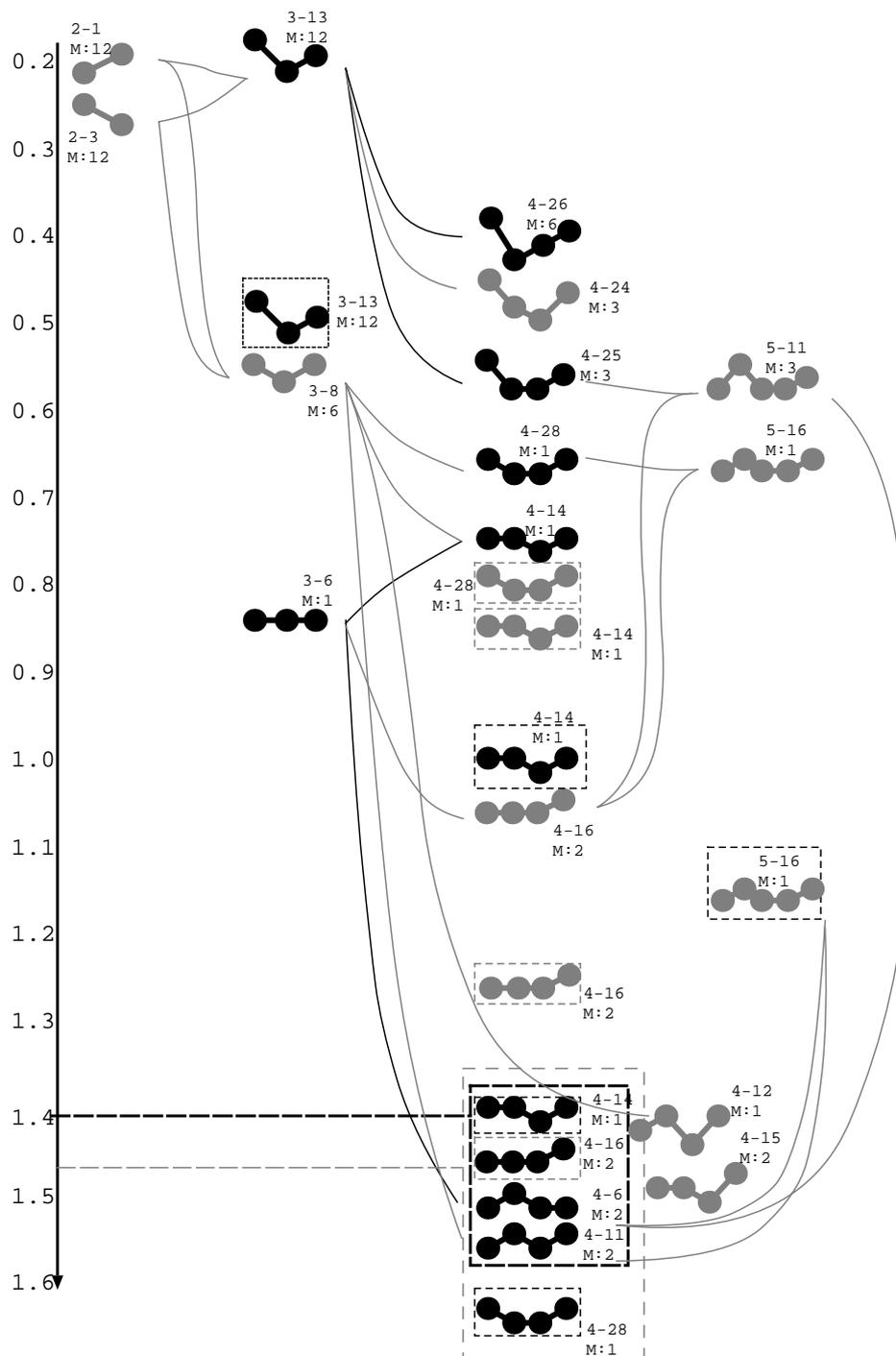


Figure 9. This graphic shows the motivic evolution tree of the 8-tone main theme of *Kunst der Fuge*. When looking from top to bottom, i.e. when our similarity allowance to identify premotives is growing, we view (in black) the evolution of the score’s motif shape obtained from our motivic analysis. For a detailed description, see the text.

This entire structure is called the *motivic evolution tree* (MET) associated with the 8-tone theme in Bach’s *Kunst der Fuge* and with the translation group TR ; we denote it by $MET(8\text{-Theme}, TR)$. It gives a global view on the motivic weight evolution and on the topologically

the tree $MET(8\text{-Theme}, ACP)$ with the full counterpoint group instead of the translation group. As to the trees of the 12-tone theme, we again use the set of all premotives of cardinality 2 to 8, as it was already done for the 8-tone theme. This yields the tree $MET(12\text{-Theme}, TR)$ and the tree $MET(12\text{-Theme}, ACP)$.

In order to keep the discussion concise, we restrict to the comparison of the trees for the full counterpoint group, i.e., $MET(8\text{-Theme}, ACP)$ and $MET(12\text{-Theme}, ACP)$. Without giving the proof here, we can assert that the comparison results for the translation group are coherent with the results that we discuss for the counterpoint group ACP .

For the following discourse, we refer to Figure 11. The motivic evolution tree $MET(8\text{-Theme}, ACP)$ (left in Figure 11) refers to the space of all premotives of the 8-tone theme, which in fact is a topological subspace of the topological space of all premotives, containing 2 to 8 tones, of the 12-tone theme. So besides the MET analysis on both METs, we are naturally provided with a further topological tool: We may ask for the topological position of premotives of the subspace, when *viewed as elements of the larger topological motivic space*. This embedding perspective is what we apply to compare the named METs.

When we look at the leading gestalts in the 12-tone theme MET (right in Figure 11), some of them are already present in the 8-tone theme MET. This is the case for gestalts 3-1, 4-2, 4-4, and 4-6. The others, i.e. 5-2, 5-4, 5-23, 5-84, 5-127, 6-53, 6-176, and 6-331 are all gestalt extensions of most of the remaining leading gestalts, i.e. 4-6, 4-13, 4-17, which live in the subspace of the 8-tone theme. Topologically, this means that the extended leading gestalts are in every neighborhood of their respectively included (leading) subspace gestalts.

7 Summary and Perspectives

Putting it all together, either a leading gestalt of the 12-tone theme is already found in the MET of the 8-tone theme, or it is contained in every neighborhood of a corresponding leading gestalt of the 8-tone theme. However, all of the extended gestalts of the 12-tone theme, except 5-2, involve tones outside the 8-tone theme. In other words, the added tones do support the motivic "substance", but topologically, this substance is "arbitrary near" to leading gestalts of the 8-tone theme.

So are we dealing with an 8-tone theme or with a 12-tone theme? The answer seems to come from a compositional remark: It is well known that the added 4-tone sequence is often used to connect adjacent parts in the overall composition. In this sense, the extension is essential from a compositional point of view: The substance of the theme, which is already present in the 8-tone theme, is connected to the last 4 tones. Therefore, these notes are connected to the substance in their role as connecting tones. So the 12-tone theme is the "right one" for compositional syntax, whereas the 8-tone theme is the right one concerning the motivic substance.

Besides this introductory discussion of an elementary example, the main theme of Bach's *Kunst der Fuge*, the present topological approach has been implemented on the RUBATO[®] software and applied to constructions of analytical performances (Mazzola G., O. Zahorka and J. Stange-Elbe (1992); Stange-Elbe J. and G. Mazzola (1998); Beran J. and G. Mazzola (2000)). Presently the RUBATO[®] software is redesigned for Mac OS X server technology by the VW research group at the Technical University of Berlin. This portation will allow more extensive

calculations, such as the motivic analysis of Arnold Schönberg's *Kammersymphonie op 9*. The motivic complexity of this work is in fact much higher than any hand-made and intuition-driven analysis can control (See Mahnkopf C.-S. (1994) for an illustration of the problem).

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Figure 11. (See next page) To the left, we show the tree $MET(8\text{-Theme}, ACP)$ and to the right, the tree $MET(12\text{-Theme}, ACP)$. Compared to the above representation in Figure 9, this one is slightly simplified: The vertical lines show the radius interval where the specific gestalt remains in leading or in second position. The horizontal connections from the 8-tone theme MET to the 12-tone theme MET indicate the gestalt inclusion relation. For a detailed MET comparison, refer to the text.

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