

Topological Motive Spaces, and Mappings of Scores’ Motivic Evolution Trees

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Abstract

This paper presents a topological model of motivic structure and analysis of music, and its extension to comparisons of motivic analyzes. Topological T_0 -spaces of motives (called *motivic spaces*), their quantification for the formalization of motives’ “*germinal function*”, and their graphs as *motivic evolution trees (MET)* form the main steps of our model that enables to broaden the concept of motif similarity to the case of motives of different cardinalities. The study of continuous functions between motivic spaces and mappings of METs introduces a model extension to comparisons of motivic analyzes. A detailed example of our approach applied to Bach’s *Art of Fugue* is presented.

1 Introduction

Various types of systematic analysis of music, such as harmonic, metric, motivic and contrapuntal, have been developed for unveiling the substance of a musical composition. Motivic analysis in particular aims at understanding a composition via its arrangement of motives. We present in this paper a topological model of motivic structure and analysis of music, and its extension about a method of analysis comparison that had been thoroughly presented in the author’s Ph.D. thesis (Buteau [4]) and also partially presented in Mazzola et. al [16]. Our deterministic approach to motivic structure follows Rudolph Reti and Kopfermann’s immanent approaches: one should determine the germinal motif of a composition by systematically comparing all motives *within* the composition. More precisely, we model the motivic analysis of music using the following definition:

Definition 1. The germinal motif is *the first appearance of a sequence of notes whose contour is “everywhere in the composition” in the sense of imitation, variation and transformation.*

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It is important to mention that our approach deals with the ‘germinal function’ of a motif, i.e., using the terminology from Definition 1, it deals with the “*everywhere in the composition*” function of motives. In particular, it enables to compute the similarity of motives of different cardinalities. Many other approaches, for example by Hörnel and Ragg [10], by Cambouropoulos [7] or by others (see [11] for an overview), easily treat motif imitations and consider also motif similarity but for only a fixed size (motif cardinality or measure window). They basically don’t consider different cardinalities, and cannot therefore present a viable approach to the “germinal function of a motif”. Indeed, many of these approaches based on the so-called (*American*) *Set Theory* (AST), which is a Music Theory using the Set Theory that has been first introduced by Allen Forte [9] and further developed by Morris[21], Lewin [12], Rahn [24], etc., can be redefined within our approach, and extended to a complete model of germinal motif. Also, a more involved approach to a mathematical motive theory was introduced by Nestke (see Nestke [22]) by using homology for measuring the complexity of motives. However, as others, Nestke does not explicitly consider germinal motives.

It is clear that dealing with different motif cardinalities considerably enhances the complexity of the model and the computations. However, despite this complexity, the model has been entirely implemented (JAVA): see Buteau [5] as an improved (and complete model implementation) version of the software module MeloRUBETTE[®] in RUBATO[®] (see the web-site [28] and Mazzola & Zahorka [19]). The latest JAVA implementation bases its computational efficiency on theorems restating the topological structure on motif classes (called *gestalts*) that significantly brings down the computational time. Also, it is important to mention regarding our model, that many other approaches are particular cases¹ in our model and investigations by use of the MeloRUBETTE[®] on Schumann’s *Träumerei* (Mazzola & Beran [1]), on Webern’s *Variation für Klavier op. 27/2* (Mazzola, Zahorka & Stange-Elbe [19]), and on Bach’s *Kunst der Fuge* (Stange-Elbe [27]) support the validity of our model.

In this self-content paper we first recall (see Buteau [2] and Mazzola et al. [16]) in Sections 2 and 3 the motivic structure of music modeled through topological T_0 -spaces, called *motivic spaces*, in which points are motives. The motivic space of a music composition is then a finite relative space of the (generic) infinite space of *all* possible motives. The crucial step in this construction is the introduction of ϵ -neighborhoods of motives, corresponding in music to variations and transformations of motives, which importantly include motif similarity for *different cardinalities*. The determination in this structure of the germinal motives is realized through our *motivic evolution tree (MET) function* representing the “*most dense*” motives in function of the (similarity threshold) neighborhood radius. The formalization of music objects and relations is carefully underlined via Formalization 3, 6, 9, 13, 20, and 22.

¹That means that by fixing some parameters— space of notes, shape type, paradigmatic group, similarity pseudo-metrics— in our model, we can easily rebuild other approaches. Importantly, we can extend them to a model of germinal motives. In other words, other approaches correspond to *partial* particular cases in our model.

In Section 4 we present a first study of continuous functions between motivic spaces. Having a topological structure on the motif level but also on the gestalt level, Theorem 24 links the function continuity on these two levels: a function between motivic spaces is continuous if and only if its corresponding function on gestalt spaces is well-defined and continuous. This leads to a natural transformation (Theorem 29) strongly linking the construct of topological spaces of motives to their corresponding spaces of gestalts. A short study of functions defined through the identity set mapping² is summarized in Theorem 28. In Section 4.3 we state corresponding results on the finite case (motivic spaces of compositions).

As a significant consequence of this categorical study we can formalize (see Formalization 32) the comparison of motivic analyzes by mapping MET's through continuous functions. In our approach, the function continuity for motivic spaces is interpreted (Lemma 33) as a sufficient and necessary condition for comparing motivic analyzes of compositions. At last we exemplify (Buteau [5]) in Section 6 our model on Bach's *Art of Fugue* for which we present all details leading to our conclusion on the length (8 or 12?) of its main theme: *the extension to twelve tones is "substantial", but it is not a proper extension* (Buteau & Mazzola [6]).

2 Motivic Topology of a Music Composition

The purpose of this section is to elaborate the basis of the mathematical model of structures that can be associated with musical objects handled in motivic analysis. For more details of this construction see Buteau [4] and Mazzola et al. [16, Chapter 22], for a detailed example of this whole construction in the context of the AST, see Buteau & Mazzola [6], and for details on the model implementation (JAVA) see Buteau [5], [4].

2.1 Motif

We restrict our attention to a minimal parameter setup in order to make the essential facts clear, and we refer the reader to the end of this section for a simple example leading to a motivic topology. First we consider the space $\mathbb{R}^{\{O,P,L,D,G,C\}} \cong \mathbb{R}^6$ of tone parametrization for which the parameters are respectively *onset*, *pitch*, *loudness*, *duration*, *glissando*, and *crescendo*. We denote the space $\mathbb{R}^{OP\dots} \subset \mathbb{R}^{\{O,P,L,D,G,C\}}$ the **space of notes** parametrized by at least onset and pitch parameters, and we consider the canonical projection $P_O : \mathbb{R}^{OP\dots} \rightarrow \mathbb{R}^{\{O\}}$ on the axis of onset events.

Definition 2. (Mazzola et al. [16], Buteau [2]) *Given a space $\mathbb{R}^{OP\dots}$ of notes, a **motif** (in $\mathbb{R}^{OP\dots}$) is a non-empty finite subset $M \subset \mathbb{R}^{OP\dots}$ such that the projection P_O canonically induces a bijection $p_O : M \rightarrow P_O(M)$. A **submotif** of a motif M is a motif M^* such that $M^* \subset M$. The **cardinality of a motif** M is the set cardinality $|M|$ of M .*

²Different topological structures are introduced on the set of motives, and the functions on gestalts induced from the motif set identity mapping, are not even always well-defined.

We fix a space $\mathbb{R}^{OP\dots}$ of notes and set $MOT(\mathbb{R}^{OP\dots}) = MOT := \{M \mid M \text{ is a motif in } \mathbb{R}^{OP\dots}\}$, and we have $MOT = \coprod_n MOT_n$ where $MOT_n = \{M \in MOT \mid M \text{ is a motif such that } |M| = n\}$.³

Intuitively, a motif is a set of tones in which only one tone occurs at a given onset time, and in which tones are not necessarily consecutive in the given composition. Motives are not necessarily germs of a musical composition, but only a priori candidates for carrying such a motivic meaning:

Formalization 3. Motives in Definition 2 are the formalization of sequences of notes, in Definition 1, from which one motif is the germinal one.

2.2 Shape Type

From now on we fix a space of notes, and introduce a generic mapping on the motives that underlines some motives' relevant contour features.

Definition 4. (Mazzola et al. [16], Buteau [2]) A **shape type** t is a family $\{\Gamma_{t,n}\}_{n \in \mathbb{N}}$ of non-empty sets $\Gamma_{t,n}$ together with a mapping

$$\begin{aligned} t : MOT &\rightarrow \Gamma_t := \bigcup_{n \in \mathbb{N}_+} \Gamma_{t,n} \\ M &\mapsto t(M) \end{aligned}$$

such that for each $n \in \mathbb{N}_+$ and $M \in MOT_n$, we have $t(M) \in \Gamma_{t,n}$. And we have the following restriction map $t_n := t|_{MOT_n} : MOT_n \rightarrow \Gamma_{t,n} : M \mapsto t_n(M)$

The set Γ_t is called a **t -space**, an element of Γ_t is called an **abstract motif (of type t)**, and $t(M)$ is called the **abstract image of motif M** . We assume⁴ in the following that $\Gamma_t := \coprod_{n \in \mathbb{N}_+} \Gamma_{t,n}$, and for each $b \in \Gamma_{t,n}$ we call n the **cardinality** of b .

Example 5. The COM-Matrix shape type represents the diastematic movement (that is the melodic contour) within the motif, i.e. between all notes in the motif. For each motif $M = \{q_1, \dots, q_n\} \in MOT_n$ we use the notation $q_i = (o_i, p_i, \dots) \in \mathbb{R}^{OP\dots}$ and assume that $o_1 < o_2 < \dots < o_n$. We consider the mapping

$$\begin{aligned} COM_n : MOT_n &\longrightarrow \{-1, 0, 1\}^{n \times n} \\ M &\longmapsto (\delta_{ij})_{ij} \end{aligned}$$

³In the classical Mathematical Music Theory (Mazzola [13]), a motif is a local composition (K, N) for which the left R -module N (R being a commutative ring with unity) is a space $\mathbb{R}^{OP\dots}$ of notes and which satisfies a bijection condition as in the above definition. In this paper, we decide to present the model with Definition 2, but one can naturally redefine the following construction of motivic spaces in the classical local composition setup as well as in the topos oriented local compositions (Mazzola et al. [16], Chapter 22).

⁴The assumption on the disjoint union of $\Gamma_{t,n}$'s simplifies the model construction but it also excludes classical examples of shape types such as the toroidal type (Mazzola [13]) which is defined through the mapping of a note into its pitch class (e.g. modulo 12) and into its onset class (e.g. modulo 12). Indeed this shape type leads to a different and interesting topological structure called specialization topology (Mazzola [13], Buteau [2]).

where $\delta_{ij} = 1$ if $p_j - p_i > 0$, 0 if $p_j = p_i$, and -1 if $p_j - p_i < 0$. This matrix $(\delta_{ij})_{ij}$ is antisymmetric and has zeros in its diagonal. For $n \geq 2$, we consider also

$$\begin{aligned} UT r_n : \{-1, 0, 1\}^{n \times n} &\longrightarrow \{-1, 0, 1\}^{n(n-1)/2} \\ (b_{ij})_{ij} &\longmapsto (b_{12}, b_{13}, \dots, b_{1n}, b_{23}, \dots, b_{(n-1)n}) \end{aligned}$$

which means that the image $UT r_n((b_{ij})_{ij})$ is the the upper triangle values of the matrix $(b_{ij})_{ij}$. Let Com_1 be defined as the unique mapping $MOT_1 \rightarrow \{\infty\}$, and for $n \geq 2$ we define the mapping

$$\begin{aligned} Com_n : MOT_n &\longrightarrow \{-1, 0, 1\}^{n(n-1)/2} \\ M &\longmapsto Com_n(M) := UT r_n \circ COM_n. \end{aligned}$$

The family $\Gamma_{Com} = \{\infty\} \cup \{\{-1, 0, 1\}^{n(n-1)/2}\}_{n \geq 2}$ together with the mapping $Com = \coprod_n Com_n$ defines the *COM-matrix shape type*.

We roughly present other shape types through the mapping t_n from which one can precisely define, as in Buteau [2] and Mazzola et al. [16], the shape type. The canonical projection $Pr_{\{O,P\}} : \mathbb{R}^{OP\dots} \rightarrow \mathbb{R}^{\{O,P\}}$ induces a mapping $M = \{q_1, \dots, q_n\} \mapsto Rg_n(M) := (Pr_{\{O,P\}}(q_i))_{i=1,\dots,n}$ with $o_1 < \dots < o_n$, and we call the induced shape $t = Rg$ the *rigid shape type*. Similarly defined, the *metric shape type* is the onset vector: $Me_n(M) := (o_1, \dots, o_n)$. The *delta-rigid shape type* corresponds to the pitch interval vector: $\Delta Rg_n(M) := (p_2 - p_1, \dots, p_n - p_{n-1})$ whereas the *delta-metric shape type* corresponds to the onset interval vector: $\Delta Me_n(M) := (o_2 - o_1, \dots, o_n - o_{n-1})$. The *rhythmical shape type* is the normalized onset interval ratios: $Rhy_n(M) := (\frac{o_2 - o_1}{L_O}, \dots, \frac{o_n - o_{n-1}}{L_O})$ with $L_O = o_n - o_1$. The *elastic shape type* is $El_n(M) := (\alpha_1, \dots, \alpha_{n-1}, r_1, \dots, r_{n-1})$ where α_i is the slope angle (radian) of $\overrightarrow{q_{(i-1)}q_i}$, $2 \leq i \leq n$, with respect to the O -axis, and $r_i = l_i/L(M)$ is the ratio of l_i , the Euclidean length of $\overrightarrow{q_{i-1}q_i}$ in the real vector space \mathbb{R}^2 , over the length $L(M) := \sum_{i=1}^{n-1} l_i$. Similarly to the COM-shape type we have the *diastematic index shape type* corresponding to the diastematic movement between consecutive notes: $India_n(M) := (b_{1,2}, b_{2,3}, \dots, b_{(n-1),n})$ with notation as in Example 5.

Formalization 6. *The abstract image of a motif in Definition 4 corresponds, in Definition 1, to the contour of a sequence of notes.*

2.3 Gestalt

We fix a shape type t , and consider a left action $\pi_t^P : P \times \Gamma_t \rightarrow \Gamma_t$ of a group P on Γ_t such that each space $\Gamma_{t,n}$ is P -invariant, i.e., $\forall n \in \mathbb{N}_+, \forall p \in P, \forall b \in \Gamma_{t,n} : p \cdot b \in \Gamma_{t,n}$. We denote⁵ this action (P, π_t^P) and call it a **paradigm for t** (Mazzola et al[16], Buteau[4]). We call the group P a **paradigmatic group (of (P, π_t^P))**. If the paradigmatic group P also acts from the left on motives $\mu_P : P \times MOT \rightarrow MOT$ and such that the two

⁵We observe that the above notation is superfluous but it however enlightens the manipulation of paradigms in Section 4 when considering two paradigms for two shape types (possibly the same).

actions μ_P and π_t^P are equivariant with respect to the shape type mapping, i.e., $\forall p \in P$, $\forall M \in MOT : p \cdot t(M) = t(p \cdot M)$, we then call the paradigm **equivariant** (Mazzola [16, Section 22.4], Buteau [4]) and denote it (P, μ_P, π_t^P) .

In the MaMuTh⁶, paradigms are often identified [16, Section 22.4] by their paradigmatic groups. It is clear that if we want to introduce a paradigm for a shape type t , we have to give first a group P , and second an action π_t^P of that group on Γ_t satisfying the above mentioned condition. For the following, we may as well identify the paradigm (P, π_t^P) with its corresponding paradigmatic group P .

Usually, equivariant actions are defined when P is a subgroup of $\overrightarrow{GL}(\mathbb{R}^{\{O, P, \dots\}})$ (the group of affine \mathbb{R} -automorphisms of the fixed vector space $\mathbb{R}^{\{O, P, \dots\}}$), acting point-wise on the motives $M \in MOT$, i.e. $p \cdot M := \{p \cdot x, x \in M\}$ for $p \in P$.

Definition 7. (Mazzola et al. [16], Buteau [2]) *Given a shape type t and a paradigm P , the $((t, P)$ -) **gestalt of a motif** M is $Ges_t^P(M) := t^{-1}(P \cdot t(M))$. For two motives M and N if $Ges_t^P(M) = Ges_t^P(N)$, then we say that M and N **have same gestalt** and write $M \sim_{Ges_t^P} N$, for which “ $\sim_{Ges_t^P}$ ” is in fact an equivalence relation (Buteau [2]) on MOT . We call the surjective mapping*

$$\begin{aligned} Ges_t^P : MOT &\twoheadrightarrow MOT / \sim_{Ges_t^P} \\ M &\mapsto Ges_t^P(M) \end{aligned}$$

the (t, P) -**gestalt mapping**.

We denote $GES_t^P := Ges_t^P(MOT) = MOT / \sim_{Ges_t^P}$ the set of all (t, P) -gestalts and $GES_{k,t}^P := Ges_t^P(MOT_k)$. It is clear that $GES_t^P = \coprod_k GES_{k,t}^P$, and for each $k \in \mathbb{N}$ and each $G \in GES_{k,t}^P$ we call $card_t^P(G) := k$ the $((t, P)$ -) **cardinality of gestalt** G .

Example 8. Let CP be the affine Klein group within $\overrightarrow{GL}(\mathbb{R}^{\{O, P, \dots\}})$. The group CP is defined as the subgroup of $\overrightarrow{GL}(\mathbb{R}^{\{O, P, \dots\}})$ generated by the subgroup of translations in onset and pitch directions, and by the linear Klein group $LCP = \langle U, K \rangle$ generated by the pitch inversion $U: U(o, p, \dots) = (o, -p, \dots)$, and the retrograde $K: K(o, p, \dots) = (-o, p, \dots)$. The group CP is a well-known transformation group (formed by transpositions, inversions, and retrograde) in European counterpoint. This is why it is called the **counterpoint group** in MaMuTh (Mazzola [13]). The group CP acts point-wise on $MOT(\mathbb{R}^{\{O, P, \dots\}})$, since translations, inversion, and retrograde transform motives into motives. In fact CP acts equivariantly for all shape types $t = Rg, Me, \Delta Rg, \Delta Me, Rhy, Com, Dia, India$ and El .

Formalization 9. *The gestalt of a motif in Definition 7 corresponds, in Definition 1, to the imitations of a sequence of notes.*

⁶Mathematical Music Theory: see Mazzola et. al [16].

2.4 Distance on Motives and on Gestalts

Given a shape type t we consider for each $n \in \mathbb{N}$ a pseudo-metric d_n on $\Gamma_{t,n}$ and set $d_{t,n}(M, N) := d_n(t(M), t(N))$ for any two motives $M, N \in MOT_n$. We denote $d = (d_n)_{n \in \mathbb{N}_+}$ and $d_t := (d_{t,n})_{n \in \mathbb{N}_+}$ and say that they are pseudo-metrics on respectively Γ_t and MOT , the latter being called a **(t -)distance on MOT** (Mazzola et al. [16], Buteau [2]).

Definition 10. *Given an equivariant paradigmatic group P for t , if for all $n \in \mathbb{N}$, all $p \in P$ and all $a, b \in \Gamma_{t,n}$, $d_n(p \cdot a, p \cdot b) = d_n(a, b)$, we say that P consists of **isometries with respect to d on space Γ_t** , and we have a pseudo-metric (Buteau [2]) on $GES_{t,k}^P$: for each $k \in \mathbb{N}$ we set*

$$Gd_{t,k}^P(G_1, G_2) := \inf_{p \in P} d_{t,k}(p \cdot M_1, M_2)$$

where $G_1, G_2 \in GES_{t,k}^P$ and $M_1, M_2 \in MOT_k$ for which $M_1 \in G_1$ and $M_2 \in G_2$. For any two motives $M_1, M_2 \in MOT_k$ we set the **gestalt distance between motives M_1 and M_2** as $gd_{t,k}^P(M_1, M_2) := d_{G,k}^P(Ges_t(M_1), Ges_t(M_2))$ (which is also a pseudo-metric on MOT_k for any $k \in \mathbb{N}$). Then we say again that d_G (resp. gd_t) is a pseudo-metric on GES_t^P (resp. on MOT). If there is no possible confusion we omit in the notation the abstract cardinality index n of d_n , $d_{t,n}$, $gd_{t,n}^P$, and of $Gd_{t,n}^P$.

Example 11. We can for example use the Euclidean metric Ed_n on $\mathbb{R}^{(n-1)n/2} \supset \Gamma_{Com,n}$, whenever $n \geq 2$, and for $n = 1$ we set $Ed_1 = 0$. This defines a *Com-distance* Ed_{Com} , called the (*Com-*)Euclidean distance, on MOT . Similarly, we can define a pseudo-metric $REd_n := Ed_n/k(n)$, where $k(n) = n(n-1)/2$ is the dimension of $\Gamma_{Com,n}$, and for which REd is called the relative *Com-Euclidean distance*, on MOT .

Each $(MOT_n, gd_{t,n}^P)$ and $(GES_{t,n}^P, Gd_{t,n}^P)$ is a pseudo-metric space. The next definition is a crucial step in our model: it links all spaces $(MOT_n, gd_{t,n}^P)$'s, respectively all $(GES_{t,n}^P, Gd_{t,n}^P)$'s, together.

Definition 12. (Mazzola et al. [16], Buteau [2]) *Given a shape type t , a pseudo-metric d on Γ_t , and a paradigmatic group P of isometries with respect to d , and such that t is P -equivariant, let $\epsilon > 0$. Then*

$$V_\epsilon(M) = V_\epsilon^{t,P,d}(M) := \{N \in MOT \mid \exists N^* \subset N \text{ s.t. } gd_t^P(M, N^*) < \epsilon\}$$

is called the ϵ -neighborhood of the motif $M \in MOT$ (with respect to t, P, d).

Formalization 13. *The variations and transformations of a sequence of notes in Definition 1 is formalized through Definition 12 as the following: a motif M is a variation of N if $N \in V_\epsilon(M)$ or if $M \in V_\epsilon(N)$ for a given $\epsilon > 0$, and a motif M is a transformation of N if $N \in V_{\epsilon'}(M)$ or if $M \in V_{\epsilon'}(N)$ for a given $\epsilon' > \epsilon$. In this formalization, the radius ϵ' relates to a similarity threshold.*

It is important to observe that our formalization of motif variation and transformation concepts includes the possibility of additional or missing notes in the variation or transformation. Also, it is clear that we need to consider both neighborhood relations, 'being in the ϵ -neighborhood' and 'containing in its ϵ -neighborhood', in order to formalize variations and transformations with Definition 12.

2.5 Motivic Topologies

To complete our formalization of the germinal motif of a musical composition, we want to enrich the sets MOT and GES_t^P with a topological structure for which the ϵ -neighborhoods should form an open basis. This is however not automatic, since it is not true in general that the intersection of ϵ -neighborhoods is a union of such neighborhoods. We then need to impose a condition on all motives 'living' together within an ϵ -neighborhood.

Definition 14. *Given a shape type t and a pseudo-metric d_t on MOT , if for any $M \in MOT_n$, $n \in \mathbb{N}_+$, any submotif $M^* \subset M$ and $\epsilon > 0$, there exists $\delta > 0$ such that for any $N \in MOT_n$: $d_t(M, N) < \delta \Rightarrow \exists$ submotif $N^* \subset N$ s.t. $card_t(N^*) = card_t(M^*)$ and $d_t(M^*, N^*) < \epsilon$, then we say that d_t is **(t -)inherited**.*

If a paradigmatic group P consists of isometries with respect to d and t is P -equivariant, then if d_t is inherited, gd_t^P is also inherited (Buteau [2]).

Example 15. For the rigid, metric, delta-rigid, delta-metric, rhythmical, diastematic, Com-Matrix, and elastic shape types the Euclidean and the relative Euclidean distances on MOT are inherited. However the distances Ed_{India} and REd_{India} for the diastematic index shape type is not inherited.

Intuitively the inheritance property insures us that two similar motives are similar *in their submotives*⁷, and is actually sufficient to insure us that the ϵ -neighborhoods form a base for a topology on MOT :

Proposition 16. (Buteau [2], Mazzola et al. [16]) *Given a shape type t , a pseudo-metric d on Γ_t , and a paradigmatic group P of isometries with respect to d , and such that t is P -equivariant, if d_t is inherited, then the collection $\{V_\epsilon(M) | M \in MOT, \epsilon > 0\}$ of all ϵ -neighborhoods forms a base for a topology $\mathcal{T}_{t,d}^P$ on MOT .*

*We call $(MOT, \mathcal{T}_{t,d}^P)$ a **motivic space**, $\mathcal{T}_{t,d}^P$ a **motivic topology (with respect to t , P , and d) for MOT** , and the quotient space $(GES_t^P, \mathcal{GT}_d)$ with topology \mathcal{GT}_d (or also denoted $\mathcal{GT}_{t,d}^P$) relative to the gestalt mapping Ges_t^P and to $\mathcal{T}_{t,d}^P$ is called the **motivic gestalt space**.*

⁷For example, a metric on Γ_{India} cannot be *India*-inherited since *India* disregards the submotif information with respect to the vector's (*India*-abstract motif) total information: what could we say, for example, of the (abstract) submotif composed with first and last tone of the motif, for which its *India*-abstract motif is $(1, -1)$? That could be (1) , (0) or even (-1) !

We reconstruct the motivic gestalt topology on gestalts by introducing on these motif classes a relation corresponding to the “submotif” relation: given a shape type t and a paradigmatic group P we say that a gestalt G^* is a **small gestalt** (Buteau [2], Mazzola et al. [16]) of the gestalt G , denoted by $G^* \sqsubset G$, if there exist motives $M^* \in G^*$ and $M \in G$ such that $M^* \subset M$. If the property holds for any triple M, M^* , and $M_1 \in MOT$ with $M^* \subset M$ and $M_1 \in Ges_t(M)$, and such that there exists a submotif M_1^* of M_1 such that $M_1^* \in Ges_t(M^*)$, then we say that gestalts **behave well (for t)** and the small gestalt relation is a transitive relation. If the pseudo-metric Gd_t^P is a metric on GES_t^P and d_t is t -inherited, then gestalts behave well for t (Buteau [2]), and we have the following:

Theorem 17. (Buteau[2], Mazzola et al.[16]) *Given a motivic gestalt topology \mathcal{GT}_d on GES_t^P as described in Proposition 16, if Gd_t^P is a metric on GES_t^P and d_t is t -inherited, then Ges_t^P is an open mapping and the collection of all sets*

$$U_\epsilon(H) := \{G \in MOT / \sim_{Ges} \mid \exists G^* \sqsubset G \text{ s.t. } Gd_t^P(G^*, H) < \epsilon\}$$

where $H \in GES_t^P$ and $\epsilon > 0$, forms an open base for \mathcal{GT}_d .

PROOF: We first observe that for a given $M_1 \in MOT$ and $\epsilon > 0$, by the definition of the relation “ \sqsubset ” and the U ’s, we have $Ges_t^P(V_\epsilon(M_1)) = U_\epsilon(Ges_t^P(M_1))$. We now claim that $Ges_t^P^{-1}(Ges_t^P(V_\epsilon(M_1))) = V_\epsilon(M_1)$ for all $\epsilon > 0$ and $M_1 \in MOT$. It is clear that $V_\epsilon(M_1) \subset Ges_t^P^{-1}(Ges_t^P(V_\epsilon(M_1)))$. For the other inclusion, suppose that $M_2 \in Ges_t^P^{-1}(Ges_t^P(V_\epsilon(M_1)))$. Then we have $Ges_t^P(M_2) = Ges_t^P(M)$ where $M \in V_\epsilon(M_1)$. By definition, this means that there exists $M^* \subset M$ such that $gd_t(M^*, M_1) < \epsilon$. Since gestalts behave well, there exists $M_2^* \subset M_2$ such that $Ges_t^P(M_2^*) = Ges_t^P(M^*)$, and $gd_t(M_2^*, M_1) = gd_t(M^*, M_1) < \epsilon$. Hence, $M_2 \in V_\epsilon(M_1)$. ■

2.5.1 Motivic Topology of a Score

We have obtained a topological structure on MOT (and on GES_t^P), the set of all possible motives. We recall that we want to model the motivic analysis of musical composition of immanent character, which means that we should exclusively deal with motives taken in the composition. This motivates the last step of our topological construction.

Definition 18. *Given a motivic space $(MOT, \mathcal{T}_{t,d}^P)$ a **motivic composition space** (MOT^*, \mathcal{T}_t^*) (with respect to $(MOT, \mathcal{T}_{t,d}^P)$) is a finite non-empty subset MOT^* of MOT , satisfying the **Submotif Existence Axiom**: given a $n_{min} \in \mathbb{N}_+$, every submotif $M^* \subset M$ of any motif $M \in MOT^*$ with $|M^*| \geq n_{min}$ is a motif within MOT^* . The relative topology of $\mathcal{T}_{t,d}^P$ on MOT^* is denoted by \mathcal{T}_t^* . The corresponding **gestalt composition space** $(GES^*, \mathcal{GT}_t^*)$ is the space $GES^* := Ges_t^P(MOT^*)$ together with the quotient topology \mathcal{GT}_t^* relative to the gestalt mapping Ges_t^P (restricted to MOT^*) and to \mathcal{T}_t^* . The **motivic composition space** $(MOT^*(S), \mathcal{T}_t^*)$ of a Score S is a motivic compo-*

sition space where $MOT^*(S)$ is a collection of motives in the musical composition S ⁸. Similarly, we have $(GES^*(S), \mathcal{GT}_t^*)$.

Given a gestalt $G \in GES^*$, the **multiplicity of G** , denoted by $mult_{t,P,MOT^*}(G)$ (or simply $mult(G)$), is the set cardinality of $\{M \in MOT^* \mid M \in G\}$. We denote $GES_k^* := GES_k \cap GES^*$, and for each $G \in GES^*$ and $\epsilon > 0$, we have $U_\epsilon^*(G) := U_\epsilon(G) \cap GES^*$.

2.5.2 Some Properties of Motivic Topologies

We state some topological properties of motivic topologies whereas all details can be seen in Buteau [2]. A motivic gestalt space (GES, \mathcal{GT}_d) is a T_0 -space and “almost T_1 -space”, in the sense that given two gestalts $G, H \in GES$ such that $G \not\sqsubset H$, there exists an open neighborhood of G which does not contain H . But if $G \sqsubset H$, then $H \in U_\epsilon(G)$ for all $\epsilon > 0$: this is exactly when GES fails the T_1 -space condition, and in particular, it shows that GES is not Hausdorff (T_2 -space). Indeed, if all translations in time are elements of the paradigmatic group P , then the topological space (GES, \mathcal{GT}_d) is irreducible, i.e. every non-empty open set in GES_t^P is dense. Also, $\overline{\{G\}} = \{H \in GES \mid H \sqsubset G\}$ and $\overline{\{G\}} = G \Leftrightarrow G \in GES_1$. Moreover GES satisfies the first axiom of countability, and if GES_1 is composed of a finite collection of gestalts then it is clear that GES is compact.

A gestalt composition space (GES^*, \mathcal{T}_t^*) is T_0 , “almost T_1 ”, and is Hausdorff space if and only if it contains only motives with same cardinality. It is also compact, and satisfies the second axiom of countability since GES^* is finite by hypothesis. If we denote $G \in GES_{n_{max}}^*$ where $n_{max} = \max\{card_t(G) \mid G \in GES^*\}$, then $\overline{\{G\}}^{\mathcal{T}_t^*}$ is an irreducible component of GES^* . Moreover GES^* is sober, i.e. each irreducible component contains one and only one generic gestalt.

3 The Problem of Germinal Motif in Motivic Spaces

Considering Formalization 13 with our topological structures we would like to say now that the germinal motif (Definition 1) corresponds to the ‘most dense’ motif in the motivic space of the musical composition. However, the elements of the open base $\{U_\epsilon(G)\}_{G \in GES^*, \epsilon > 0}$ and the only ‘almost T_1 ’ space property show us that the geometry of our spaces does not correspond to a standard Euclidean space, and this motivates our construction of the following quantification functions.

First we consider two gestalts G and $H \in GES^* = GES^*(MOT^*)$ and a neighborhood radius $\epsilon > 0$. We measure the *presence of gestalt G in gestalt H* (or, with inverse roles: *H being “contained” in G*) at ϵ by the following intensity number: let $M_G \in G$ and $M_H \in H$, then

$$Int_{\epsilon,G}(H) := card\{M_H^* \subset M_G \mid card(M_H^*) = card(M_G) \wedge M_H^* \in V_\epsilon^*(M_G)\} \cdot mult(H)$$

⁸Given a musical composition S we can naturally introduce a correspondence between S and a non-empty finite set $\mathcal{S} \subset \mathbb{R}^{OP\dots}$. This intuitive correspondence can be rigorously set; For details see Mazzola et al. [16], Buteau [4].

which is well-defined (Buteau [4]) and where $\text{Int}_{\epsilon, G}(H) = 0 \Leftrightarrow H \notin U_{\epsilon}^*(G)$. Since the higher cardinality difference between G and H the higher the probability that submotives $M_H^* \subset M_H$ have with M_G the distance $gd_t^P(M_H^*, M_G) < \epsilon$, we weight the intensity by $\frac{1}{2^{(\text{card}_t(H) - \text{card}_t(G))}}$. The presence and the content of gestalt G in the whole topological space GES^* , at neighborhood radius $\epsilon > 0$, is then defined by summing up its presence (resp. content) in every gestalt of GES^* :

Definition 19. (Mazzola [16], Buteau [2]) *Given a gestalt composition space GES^* , a gestalt $G \in GES^*$, and $\epsilon > 0$, then the **presence of G (at radius ϵ)**, respectively the **content of G (at radius ϵ)**, is*

$$\begin{aligned} \text{Presence}(G, \epsilon) &:= \sum_{H \in GES^*} \frac{1}{2^{(\text{card}_t(H) - \text{card}_t(G))}} \cdot \text{Int}_{\epsilon, G}(H) \\ \text{Content}(G, \epsilon) &:= \sum_{H \in GES^*} \frac{1}{2^{(\text{card}_t(G) - \text{card}_t(H))}} \cdot \text{Int}_{\epsilon, H}(G), \end{aligned}$$

respectively. The **weight of G (at radius ϵ)** is

$$\text{Weight}(G, \epsilon) := \text{Presence}(G, \epsilon) \cdot \text{Content}(G, \epsilon).$$

We call these three functions respectively **presence**, **content** and **weight functions**.

The weight of a gestalt was introduced in order to obtain a ‘global’ topological information about the gestalt since *Presence* and *Content* quantify an ‘opposite’ motivic information. Clearly, this quantification (presence, content, weight) is not unique; see Buteau [4],[3] for a generalization of these functions.

Formalization 20. *Given a similarity threshold ϵ_{max} , a germinal motif in Definition 1 corresponds to the first motif appearance M in the musical composition with heaviest gestalt H at radius ϵ_{max} : $\text{Weight}(H, \epsilon_{max}) \geq \text{Weight}(G, \epsilon_{max})$ for all $G \in GES^*$, and $M \in H$ with $M < N$ for all $N \in H$ where ‘ $<$ ’ is the temporal ordering.*

The determination in our mathematical structure of the germinal motif involves the choice of the neighborhood radius corresponding to, as explained at Formalization 13, a similarity threshold. This choice of ‘adequate’ radius is in the domain of cognitive sciences, and therefore we don’t try to fix a radius, but instead we decide to consider the germinal motif in function of the neighborhood radius: we consider the homeomorphism (Mazzola et al. [16]) $\phi^S : GES^* \rightarrow GES^{*S} : G \mapsto \overline{G} := \overline{\{G\}}$ for which GES^{*S} is the sober space associated to GES^* , we fix a **tolerance** $n \in \mathbb{N}$, and set

$$\begin{aligned} \text{heavy}_n : GES^{*S} \times \mathbb{R}_+ &\rightarrow [0, 1] \\ (\overline{G}, \epsilon) &\mapsto \begin{cases} 1/n & \text{if } G \text{ is the } n\text{th-heaviest gestalt} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

from which we consider the fiber Θ of $(0, 1]$.

Definition 21. *We call the function heavy_n restricted to Θ the **Motivic Evolution Tree Function Met_n of GES^*** (with respect to MOT^* , t , P , and d) and its graph, the*

Motivic Evolution Tree (MET). *The following commutative diagram, where inj is the canonical inclusion mapping, shows the construction of the Met_n function:*

$$\begin{array}{ccc}
 GES^{*S} \times \mathbb{R}_+ & & \\
 \downarrow \text{heavy}_n & \swarrow \text{inj} & \\
 & & \Theta \\
 & \searrow \text{Met}_n & \\
 [0, 1] & &
 \end{array}$$

We refer the reader to Section 6.1 for a detailed example of a MET.

Formalization 22. *The resulting motivic analysis of a musical composition S corresponds to the MET of $GES^*(S)$ (with respect to a segmentation $MOT^*(S)$ and to a choice of topological parameters t , P , and d).*

Remark 23. The gestalt weight function quantifies the topological structure of the gestalt composition space of a score and we expect that the graph of the latter function should represent the essence of the motivic structure extracted from the topological space. However a central concept, the very proximity shared between a gestalt and a small gestalt, in our topological construction had disappeared through the quantification, and it had to be present in any final graphic representing the topological hierarchy of a gestalt composition space.

We recall that if a gestalt G is a small gestalt of H , then H is in all open sets around G . Intuitively, in music theory, this also means that a motif is strongly related (or very close) to its submotives. In the gestalt composition space, the small gestalt relation can also be restated through the closure operation: $G \sqsubset H \Leftrightarrow G \in \overline{\{H\}}$. And this motivated our consideration of the canonical homeomorphism ϕ^S between a gestalt composition space and its associated sober space. In fact, if $G \sqsubset H$, then we have $\phi^S(G) = \overline{G}$ and $\phi^S(H) = \overline{H}$, but also $\overline{G} \subset \overline{H}$. We call elements of GES^{*S} **closed gestalts**.

4 Continuous Functions between Motivic Spaces

We consider a set mapping $f : MOT \rightarrow MOT$ between motives which induces, for given two motivic topologies \mathcal{T}^1 and \mathcal{T}^2 on MOT , a function

$$\begin{array}{ccc}
 f_{\mathcal{T}^1, \mathcal{T}^2}(= f) : & (MOT, \mathcal{T}^1) & \longrightarrow & (MOT, \mathcal{T}^2) \\
 & M & \longmapsto & f(M)
 \end{array}$$

between motivic spaces. We consider also the motivic gestalt spaces (GES^i, \mathcal{GT}^i) with respect to the gestalt mappings $Ges^i : (MOT, \mathcal{T}^i) \rightarrow (GES^i, \mathcal{GT}^i)$ for $i = 1, 2$, and a

section $R : GES^1 \rightarrow MOT$ of the gestalt mapping Ges^1 . Then we have the following diagram:

$$\begin{array}{ccc} (MOT, \mathcal{T}^1) & \xrightarrow{f} & (MOT, \mathcal{T}^2) \\ R \updownarrow \scriptstyle Ges^1 & & Ges^2 \downarrow \\ (GES^1, \mathcal{GT}^1) & \xrightarrow{G_R f} & (GES^2, \mathcal{GT}^2) \end{array}$$

in which $G_R f$ is the function induced by R and f : $G_R f := Ges^2 \circ f \circ R$, called the (f, R) -**gestaltic function**. If $G_R f = G_{R'} f$ for any two sections R and R' of Ges^1 , then the above diagram is commutative, and conversely. In this case we say that $G_R f$ is well-defined, we denote it Gf and call it the f -**gestaltic function**. We have the following theorem that states properties for studying categories of motivic (gestalt) spaces.

Theorem 24. (Buteau [4]) *Let f be a function between motivic spaces as above introduced, R a section of Ges^1 , and $G_R f$ be the (f, R) -gestaltic function. Then*

i) f is continuous $\Leftrightarrow G_R f$ is well-defined and continuous.

Moreover if f is continuous, then

ii) f is surjective $\Rightarrow Gf$ is surjective;

iii) f is open $\Rightarrow Gf$ is open;

iv) f is a homeomorphism $\Rightarrow Gf$ is a homeomorphism;

v) The small gestalt relation is preserved through Gf , i.e. $\forall H^, H \in GES^1 : H^* \sqsubset H \Rightarrow Gf(H^*) \sqsubset Gf(H)$.*

Proof. *i)* Suppose first that $G_R f$ is well-defined and continuous. Let $\epsilon > 0$ and $M \in MOT$, and consider $f^{-1}(V_\epsilon^{\mathcal{T}^2}(M)) := V$. If V is empty, we are done. Otherwise let $N \in V$. First we observe that $Ges^1(N) \subset V$ since Gf is well-defined and ϵ -neighborhoods of motives are stable with respect to gestalts. Consider $U_\epsilon^{\mathcal{GT}^2}(Ges^2(M))$. Since Gf is continuous and $Gf(Ges^1(N)) = Ges^2(f(N)) \in U_\epsilon^{\mathcal{GT}^2}(Ges^2(M))$, there exists $\delta > 0$ such that $U_\delta^{\mathcal{GT}^1}(Ges^1(N)) \subset Gf^{-1}(U_\epsilon^{\mathcal{GT}^2}(Ges^2(M)))$. But then $V_\delta^{\mathcal{T}^1}(N) \subset V$: otherwise there exists a motif $N' \in V_\delta^{\mathcal{T}^1}(N)$ and $N' \notin V$, which implies that $Ges^1(N') \in U_\delta^{\mathcal{GT}^1}(Ges^1(N))$ and $Ges^1(N') \notin Gf^{-1}(U_\epsilon^{\mathcal{GT}^2}(Ges^2(M)))$, a contradiction. Therefore f is continuous.

Suppose now that f is continuous and let the gestalt $H \in GES^1$ and motives $M_1, M_2 \in (MOT, \mathcal{T}^1)$ such that $M_1, M_2 \in H$. We want to show that $Ges^2(f(M_1)) = Ges^2(f(M_2))$, i.e. $f(M_1) \sim_{Ges^2} f(M_2)$. But then if motives $f(M_1)$ and $f(M_2)$ don't have same gestalt, there exists an $\epsilon > 0$ such that w.l.o.g. $f(M_1) \notin V_\epsilon^{\mathcal{T}^2}(f(M_2))$, open set in (MOT, \mathcal{T}^2) . That implies that the motif $M_1 \notin f^{-1}(V_\epsilon^{\mathcal{T}^2}(f(M_2)))$, and since $M_1 \sim_{Ges^1} M_2$ by hypothesis ($M_1, M_2 \in H$!), this means that $f^{-1}(V_\epsilon^{\mathcal{T}^2}(f(M_2)))$ is not open, a contradiction to the continuity of f . Therefore $f(M_1) \sim_{Ges^2} f(M_2)$ and Gf is well-defined. The continuity of Gf follows from the fact that gestalt mappings are open and continuous.

ii) and iii) are straightforward.

iv) It follows from above that Gf is well-defined, continuous, open and surjective. We now show the injectivity: suppose we have two gestalts $G_1, G_2 \in GES^1$ such that $G_1 \neq G_2$. Then there are two motives $M_1, M_2 \in MOT$ with $M_1 \in G_1$, $M_2 \in G_2$, and $M_1 \not\sim_{Ges^1} M_2$. Therefore there exists an $\epsilon > 0$ such that w.l.o.g. $M_1 \notin V_\epsilon^{T^1}(M_2)$. Since f is injective, then $f(M_1) \notin f(V_\epsilon^{T^1}(M_2))$. But f is also open, so there exists a $\delta > 0$ such that $V_\delta^{T^2}(f(M_2)) \subset f(V_\epsilon^{T^1}(M_2))$. Therefore $f(M_1) \notin V_\delta^{T^2}(f(M_2))$, and this implies that $f(M_1) \not\sim_{Ges^2} f(M_2)$. Since f is continuous and therefore Gf is well-defined, this means that $Gf(G_1) \neq Gf(G_2)$.

v) Suppose that Gf is well-defined and continuous, and that $Gf(H^*) \not\sqsubset Gf(H)$ for two gestalts $H^*, H \in GES^1$. Then there exists an $\epsilon > 0$ such that $Gf(H) \notin U_\epsilon^{GT^2}(Gf(H^*))$. Since Gf is continuous there exists a $\delta > 0$ such that $U_\delta^{GT^1}(H^*) \subset Gf^{-1}(U_\epsilon^{GT^2}(Gf(H^*)))$ and $Gf^{-1}(Gf(H)) \cap U_\delta^{GT^1}(H^*) = \emptyset$, which implies that $H \notin U_\delta^{GT^1}(H^*)$ and that $H^* \not\sqsubset H$. \square

We remark that the opposite of *iii)* is in general not true. Indeed, Proposition 25 will present a continuous function $f = id$ on motivic spaces for which Gid is open, but not id . This follows from the fact that gestalt mappings are surjective but not injective. When a motif M is mapped to a motif $N \in (MOT, T^2)$, then $Ges_t^{P_1}(M)$ is mapped to $Ges_t^{P_2}(N)$ by definition, whenever the corresponding gestaltic function is well-defined. However, this does not imply that all motives $N' \in Ges_t^{P_2^{-1}}(Ges_t^{P_2}(N))$ within the gestalt of N are images of motives M' in the gestalt $Ges_t^{P_1}(M)$ of M . And this is exactly where the openness of the mapping may fail.

Also we should observe that the injectivity of Gf in *iv)* follows from the continuity, injectivity, and openness of f . Finally, the property *v)* actually follows from the fact that gestalts are ‘*topological*’, in the sense that we could redefine them through the topological structure \mathcal{T} on MOT : for any motif $M \in MOT$ and any $\epsilon > 0$: $Ges_t^P(M) = V_\epsilon^T(M) \cap \overline{\{M\}}$.

4.1 The Set Identity-Gestaltic Functions

We consider the identity set mapping $M \mapsto id(M) = M$ on motives and its induced $id_{\mathcal{T}^1, \mathcal{T}^2}(= id) : (MOT, \mathcal{T}^1) \rightarrow (MOT, \mathcal{T}^2) : M \mapsto M$ called the **set identity function (with respect to \mathcal{T}^1 and \mathcal{T}^2)**. It is bijective, and given a section R of Ges^1 , the (id, R) -gestaltic function $G_R id : GES_{t_1}^{P_1} \rightarrow GES_{t_2}^{P_2}$, if well-defined, is surjective but not necessarily injective, and id (as well as Gid) may or may not be continuous depending on the motivic spaces (MOT, \mathcal{T}^1) and (MOT, \mathcal{T}^2) .

If $G_R id$ is well-defined, i.e. $G_R id = Gid$, we observe that, independent of the motivic topologies, $\forall H^*, H \in GES_{t_1}^{P_1} : H^* \sqsubset H \Rightarrow Gid(H^*) \sqsubset Gid(H)$, and, in particular, $\forall M, N \in MOT : M \sim_{Ges_{t_1}^{P_1}} N \Rightarrow M \sim_{Ges_{t_2}^{P_2}} N$.

Since motivic spaces are defined through the three parameters t, P , and d , we study set identity functions on motivic spaces by varying each of these parameters. First, fixing t and P , we easily show (Buteau [4]) that if the metric Gd_k^P is equivalent to Gd_k^P for each

$k \in \mathbb{N}$, then \mathcal{GT}_d and $\mathcal{GT}_{d'}$ are the same topology for GES_t^P . Second, given a shape type t and an equivariant paradigm (P, μ_P, π_t^P) for t , we introduce the corresponding faithful paradigmatic group $P_{faith|t}$ for (P, μ_P, π_t^P) , i.e., $P_{faith|t} := P / \text{Ker}(P \rightarrow \text{Perm}(\Gamma_t))$.

Proposition 25. (Buteau [4]) *Given a shape type t , two equivariant paradigms $(P_1, \mu_{P_1}, \pi_t^{P_1})$ and $(P_2, \mu_{P_2}, \pi_t^{P_2})$ for t such that $P_{1\,faith|t}$ is a subgroup of $P_{2\,faith|t}$ for which the group action $\mu_{P_1\,faith}$ is canonically induced by the action μ_{P_2} , and a pseudo-metric d for t such that either $\mathcal{T}_{t,d}^{P_2}$ is one of the topologies defined by the parameter $(t, P, \text{ and } d)$ examples in Section 2 or the index $(P_{2\,faith|t} : P_{1\,faith|t})$ is finite. Then the gestaltic function*

$$Gid_{\mathcal{T}_{t,d}^{P_1}, \mathcal{T}_{t,d}^{P_2}} : (GES, \mathcal{GT}_{t,d}^{P_1}) \rightarrow (GES, \mathcal{GT}_{t,d}^{P_2})$$

is open and continuous.

Moreover Gid is a homeomorphism if and only if $P_{1\,faith|t} = P_{2\,faith|t}$, i.e., only if $GES_t^{P_2} = GES_t^{P_1}$ and Gid is the identity mapping on gestalts.

Proof. Since $Gid_{\mathcal{T}_{t,d}^{P_1}, \mathcal{T}_{t,d}^{P_1\,faith|t}}$ and $Gid_{\mathcal{T}_{t,d}^{P_2}, \mathcal{T}_{t,d}^{P_2\,faith|t}}$ are isomorphisms we suppose w.l.o.g. that $P_i = P_{i\,faith|t}$, $i = 1, 2$ and denote $id = id_{\mathcal{T}_{t,d}^{P_1}, \mathcal{T}_{t,d}^{P_2}}$. To prove that the function Gid is continuous we claim that $\forall \epsilon > 0$ and $\forall G \in GES_t^{P_2}$:

$$Gid^{-1}(U_\epsilon^{\mathcal{GT}_{t,d}^{P_2}}(G)) = \bigcup_{i=1}^{(P_2:P_1)} U_\epsilon^{\mathcal{GT}_{t,d}^{P_1}}(G_i)$$

where for a fixed motif $id^{-1}(M_G) = M_G \in G$ and $\forall \bar{p}_i \in P_2/P_1$: $G_i = Ges_t^{P_1}(p_i \cdot M_G)$ for any representative $p_i \in P_2$ of \bar{p}_i .

Suppose first that the gestalt $H \in \bigcup_{i=1}^{(P_2:P_1)} U_\epsilon^{\mathcal{GT}_{t,d}^{P_1}}(G_i) \subset GES_t^{P_1}$. Then $\exists H^* \sqsubset H$ such that $Gd_1(H^*, G_i) = \inf_{g \in P_1} d(g \cdot t(M_{H^*}), t(M_{G_i})) < \epsilon$ for one i and for any motives $M_{H^*} \in H^*$ and $M_{G_i} \in G_i$. But since $P_1 \subset P_2$, we have

$$\begin{aligned} \inf_{g \in P_1} d(g \cdot t(M_{H^*}), t(M_{G_i})) &\geq \inf_{g \in P_2} d(g \cdot t(M_{H^*}), t(M_{G_i})) \\ &\stackrel{\text{Def. of } Gd}{=} Gd_2(Ges_t^{P_2}(M_{H^*}), Ges_t^{P_2}(M_{G_i})) \\ &\stackrel{\text{Def. of } Gid}{=} Gd_2(Gid(Ges_t^{P_1}(M_{H^*})), Gid(Ges_t^{P_1}(M_{G_i}))) \\ &= Gd_2(Gid(H^*), Gid(G_i)) \\ &= Gd_2(Gid(H^*), G). \end{aligned}$$

Since $Gid(H^*) \sqsubset Gid(H)$ and $Gd_2(Gid(H^*), G) < \epsilon$, then it implies that $Gid(H) \in U_\epsilon^{\mathcal{GT}_{t,d}^{P_2}}(G)$, which means that $H \in Gid^{-1}(U_\epsilon^{\mathcal{GT}_{t,d}^{P_2}}(G))$.

Suppose now that the gestalt $H \in Gid^{-1}(U_\epsilon^{\mathcal{GT}_{t,d}^{P_2}}(G)) \subset GES_t^{P_1}$. It implies that $\tilde{H} := Gid(H) \in U_\epsilon^{\mathcal{GT}_{t,d}^{P_2}}(G)$, i.e., $\exists \tilde{H}^* \sqsubset \tilde{H}$ such that $Gd_2(\tilde{H}^*, G) < \epsilon$. But $Gd_2(\tilde{H}^*, G) \stackrel{\text{Def.}}{=} \inf_{g \in P_2} d(g \cdot t(M_G), t(M_{\tilde{H}^*}))$ for any $M_G, M_{\tilde{H}^*} \in (MOT, \mathcal{T}_{t,d}^{P_2})$ with $G = Ges_t^{P_2}(M_G)$ and

Theorem 28. (Buteau [4]) *Given two motivic gestalt spaces $(GES_{t_1}^{P_1}, \mathcal{GT}_{d_1})$ and $(GES_{t_2}^{P_2}, \mathcal{GT}_{d_2})$, where t_1 and $t_2 = Rg, \Delta Rg, Me, \Delta Me, Dia, El, Com,$ or Rhy , and with respective paradigms $(P_1, \mu_{P_1}, \pi_{t_1}^{P_1})$ and $(P_2, \mu_{P_2}, \pi_{t_2}^{P_2})$ such that if we consider also the paradigm $(P_1, \mu_{P_1}, \pi_{t_2}^{P_1})$ for t_2 , $P_1 \text{ faith}|_{t_2} \subset P_2 \text{ faith}|_{t_2}$ and the group action $\mu_{P_1 \text{ faith}|_{t_2}}$ is canonically induced by the action $\mu_{P_2 \text{ faith}|_{t_2}}$. Suppose that $Gd_{1k}^{P_1}$ (resp. $Gd_{2k}^{P_2}$) is equivalent to $GED_{1k}^{P_1}$ (resp. to $GED_{2k}^{P_2}$) for all $k \in \mathbb{N}_+$, and suppose also that the set identity gestaltic function $Gid_{\mathcal{T}_{t_1, Ed_1}^{Id}, \mathcal{T}_{t_2, Ed_2}^{Id}}$ is well-defined and continuous.*

If $P_2 \text{ faith}|_{t_2} \subset CP$ or if $P_1 \text{ faith}|_{t_1}$ and $P_2 \text{ faith}|_{t_2}$ are finite groups, then the gestaltic function $Gid_{\mathcal{T}_{t_1, d_1}^{P_1}, \mathcal{T}_{t_2, d_2}^{P_2}} : GES_{t_1}^{P_1} \rightarrow GES_{t_2}^{P_2}$ is continuous.

Proof. The following diagram

$$\begin{array}{ccc}
 (GES_{t_1}^{P_1}, \mathcal{GT}_{Ed}) & \longrightarrow & (GES_{t_2}^{P_1 \text{ faith}|_{t_2}}, \mathcal{GT}_{Ed}) \\
 \updownarrow & & \downarrow \\
 & & (GES_{t_2}^{P_2}, \mathcal{GT}_{Ed}) \\
 & & \updownarrow \\
 (GES_{t_1}^{P_1}, \mathcal{GT}_{d_1}) & \longrightarrow & (GES_{t_2}^{P_2}, \mathcal{GT}_{d_2})
 \end{array}$$

where each mapping is a continuous (Buteau [4]) set identity-gestaltic function, commutes, and it follows that the gestaltic function $Gid_{\mathcal{GT}_{t_1, d_1}^{P_1}, \mathcal{GT}_{t_2, d_2}^{P_2}}$ is well-defined and continuous. \square

For other examples of continuous gestaltic functions, such as symmetry-gestaltic functions, see Buteau [4].

4.2 Gestalt Mappings as Natural Transformations

We consider the full subcategory $MotS$ of \mathbf{Top} , the category of topological spaces whose morphisms are continuous functions, for which the objects are motivic spaces (MOT, \mathcal{T}) . Also, we consider the subcategory $MotS_{Id}$ of motivic spaces for which morphisms are set identity continuous functions. All morphisms in $MotS_{Id}$ are then bijective, and therefore epimorphisms and monomorphisms, but not necessarily isomorphisms since some set identity functions are not open in general, and there is, given two motivic spaces, at most one morphism. Similarly, we have the full subcategory $GMotS$ of \mathbf{Top} whose objects are gestalt motivic spaces (GES, \mathcal{GT}) , and the category $GMotS_{Id}$ whose morphisms are set identity-gestaltic continuous functions $Gid_{\mathcal{T}^1, \mathcal{T}^2} : (GES, \mathcal{GT}^1) \rightarrow (GES, \mathcal{GT}^2)$.

We have the covariant, full, and not faithful functor G (Buteau[4]) which is defined as follows:

$$\begin{array}{ccc}
 G : & MotS & \longrightarrow & GMotS \\
 & (MOT, \mathcal{T}_{t_1, d_1}^{P_1}) & \longmapsto & (Ges_{t_1}^{P_1}(MOT), \mathcal{GT}_{d_1}) \\
 & \quad \downarrow f & & \quad \downarrow G(f) := Gf \\
 & (MOT, \mathcal{T}_{t_2, d_2}^{P_2}) & \longmapsto & (Ges_{t_2}^{P_2}(MOT), \mathcal{GT}_{d_2})
 \end{array}$$

Also we have the restriction G_{Id} of G on the subcategories $MotS_{Id}$ and $GMotS_{Id}$ defining a full and faithful functor.

Theorem 29. (Buteau [4]) *Let I be the inclusion functor from $MotS$ to \mathbf{Top} , and let $Ges : I \rightarrow G$ be defined as follows: for each motivic space $(MOT, \mathcal{T}_{t,d}^P)$ in $MotS$, let the morphism in \mathbf{Top} be $Ges_{(MOT, \mathcal{T}_{t,d}^P)} := Ges_t^P$, i.e. its corresponding gestalt mapping. Then Ges is a natural transformation. Similarly, there is a natural transformation for the inclusion functor from $MotS_{Id}$ to \mathbf{Top} .*

4.3 Categories of Motivic Composition Spaces

Similarly to the construction of f -gestaltic functions on motivic gestalt spaces, we have the g -gestaltic composition function G_{Rg} :

$$\begin{array}{ccc} (MOT_1^*, \mathcal{T}^{1*}) & \xrightarrow{g} & (MOT_2^*, \mathcal{T}^{2*}) \\ \begin{array}{c} R \uparrow \downarrow \\ \downarrow \uparrow \end{array} Ges^{1*} & & \downarrow Ges^{2*} \\ (GES^{1*}, \mathcal{GT}^{1*}) & \xrightarrow{G_{Rg}} & (GES^{2*}, \mathcal{GT}^{2*}) \end{array}$$

between gestalt composition spaces GES^{1*} and GES^{2*} and we easily observe similar properties to gestaltic functions as described in Theorem 24. As important implication of these properties, we have the following corollary:

Corollary 30. (Buteau [4]) *Given a function $g : (MOT_1^*, \mathcal{T}^{1*}) \rightarrow (MOT_2^*, \mathcal{T}^{2*})$ for the motivic composition spaces $(MOT_1^*, \mathcal{T}^{1*})$ and $(MOT_2^*, \mathcal{T}^{2*})$, then g is continuous $\Leftrightarrow Gg$ is well-defined and the small gestalt relation is preserved through Gg .*

Proof. Let $\epsilon_0^i := \min\{gd_t^{P_i}(M, N) \mid M, N \in MOT_i^* \wedge M \notin Ges^{i*}(N)\}$. Suppose that there exist $M_1 \in MOT_1^*$ and $M_2 \in MOT_2^*$ such that $V_{\epsilon_0^1}^{T^{1*}}(M_1) \not\subset g^{-1}(V_{\epsilon_0^2}^{T^{2*}}(M_2))$ where $g(M_1) = M_2$, i.e. there exists $N \in MOT_1^*$ such that $N \in V_{\epsilon_0^1}^{T^{1*}}(M_1)$ and $N \notin g^{-1}(V_{\epsilon_0^2}^{T^{2*}}(M_2))$. But this means that $Ges^{1*}(N) \supset Ges^{1*}(M_1)$ and that $Ges^{2*}(g(N)) \not\subset Ges^{2*}(M_2)$, which shows that the small gestalt relation is not preserved through Gg , and we are done. \square

As a special case of well-defined and continuous gestaltic composition functions, we naturally have restrictions of continuous functions f on motivic spaces, whenever $f(MOT_1^*) \subset f(MOT_2^*)$. For example, we can restrict continuous set identity-gestaltic functions to motivic composition spaces such that $MOT_1^* \subset MOT_2^*$. This can be leading to interesting applications in music theory such as an inquiry about the length of Bach's *Art of Fugue* main theme (8 or 12 notes?): see Section 6.2. An example of continuous gestaltic composition function is the following: we consider the gestaltic function $Gid_{\mathcal{T}_{t,Ed}^{Id}, \mathcal{T}_{Com,Ed}^{Id}}$ that is well-defined for $t = Rg, \Delta Rg, El$, or Dia but not continuous. But when restricting it to finite collections of motives, we gain the continuity of the restricted $Gid_{\mathcal{T}_{t,Ed}^{Id}, \mathcal{T}_{Com,Ed}^{Id}}^*$ function. This is a consequence of Corollary 30.

Corollary 31. (Buteau [4]) *With the hypotheses of Theorem 28 for which the shape type $t_2 = Com$ and where $Gid_{\mathcal{T}_{t,Ed}^{Id}, \mathcal{T}_{Com,Ed}^{Id}}$ is well-defined but not necessarily continuous, let $MOT_1^* \subset MOT_2^*$. Then $id_{\mathcal{T}_{t,d_1}^{P_1}, \mathcal{T}_{Com,d_2}^{P_2}} : MOT_1^* \rightarrow MOT_2^*$ is continuous, and so is $Gid_{\mathcal{T}_{t,d_1}^{P_1}, \mathcal{T}_{Com,d_2}^{P_2}}^*$.*

We mention that we could think of defining *note mappings* between two scores, which could then induce a function between their respective motivic composition spaces. In general, the gestaltic function is however not well-defined (Buteau [4]).

Finally we have analogous subcategories of topological spaces to the ones defined in Section 4.2. For this finite case we first define the subcategories Mot^*S , $MotS^*$, and $MotS_{Id}^*$ of **Top** for which the objects are all the motivic composition spaces, and for which morphisms are respectively continuous functions, well-defined functions $f^* : (MOT_1^*, \mathcal{T}^{1*}) \rightarrow (MOT_2^*, \mathcal{T}^{2*})$ which are restrictions of continuous functions $f \in mor_{MotS}((MOT, \mathcal{T}^1), (MOT, \mathcal{T}^2))$, and restrictions of continuous set identity functions. We also have their corresponding categories of gestalt composition spaces $GMot^*S$, $GMotS^*$, and also $GMotS_{Id}^*$. Moreover, we can also naturally define functors G^* , $G|^*$, and G_{Id}^* between corresponding categories, and, similarly to Section 4.2, we also introduce natural transformations with the inclusion functor I from $MotS^*$ (resp. $MotS^*$ and $MotS_{Id}^*$) to **Top**, and let $Ges^* : I \rightarrow G^*$ be the natural transformation (Buteau [4]).

We can also consider full subcategories of the previous ones by fixing a musical composition S and considering all motivic spaces $MOT^*(S)$ for S . In particular, we have the full subcategory of $MotS_{Id}^*$. The study of this category corresponds, following Formalization 20, to the search of understanding of the composition's motivic structure through the whole variety of motivic structures about that composition: it is the *Yoneda philosophy* (Mazzola et al. [16, Chapter 9]).

5 Mappings of Motivic Evolution Trees

We combine in this section the concepts of Motivic Evolution Tree (Section 3) and of continuous gestaltic composition functions (Section 4.3): through a continuous and well-defined gestaltic function on composition gestalt spaces, we can *map the corresponding MET* of the domain space, and compare the resulting *MET image* to the actual MET of the co-domain space.

Given two motivic composition spaces MOT_1^* and MOT_2^* , and a continuous function

$g : MOT_1^* \rightarrow MOT_2^*$, we consider the following diagram:

$$\begin{array}{ccc}
MOT_1^* \times \mathbb{R}_+ & \xrightarrow{(g,k)} & MOT_2^* \times \mathbb{R}_+ \\
\downarrow (\phi^S \circ Ges^{1*}, i) & & \downarrow (\phi^S \circ Ges^{2*}, i) \\
GES_1^{*S} \times \mathbb{R}_+ & \xrightarrow{(Gg^S, k)} & GES_2^{*S} \times \mathbb{R}_+ \\
\downarrow heavy_n^{GES_1^*} & \swarrow inj & \downarrow heavy_n^{GES_2^*} \\
& \Theta_{GES^1} & \Theta_{GES^2} \\
& \swarrow Met_{1/n}^{GES^1} & \swarrow Met_{1/n}^{GES^2} \\
[0, 1] & & [0, 1]
\end{array}$$

for which k is a stretching function⁹ of \mathbb{R}_+ , the function i is the identity function on \mathbb{R}_+ , $Gg^S := \phi^S \circ Gg$ is making the top diagram commute, the function $heavy_{GES_2^*}^{GES_1^*}$ is defined, for $\overline{G} \in GES_2^{*S}$ and $\epsilon > 0$, as

$$heavy_{GES_2^*}^{GES_1^*}(\overline{G}, \epsilon) := \max_{\overline{G}' \in GES_1^{*S}} \{heavy^{GES_1^*}(\overline{G}', k^{-1}(\epsilon)) \mid Gg^S(\overline{G}') = \overline{G}\},$$

with fiber $\Theta := heavy_{GES_2^*}^{GES_1^*}^{-1}((0, 1])$, $DMet_{GES^2}^{GES^1}$ is the restriction of $heavy_{GES_2^*}^{GES_1^*}$ to Θ , and inj are injection functions making the 3 lower triangular diagrams commute. We call the above diagram the **MET-comparison diagram for GES^{1*} and GES^{2*} with respect to Gg** .

We observe first that given two gestalts G_1, G_2 in GES^{1*} with $Gg(G_1) = Gg(G_2)$, it can happen that $heavy_n^{GES_1^*}(\overline{G}_1, \epsilon) \neq heavy_n^{GES_1^*}(\overline{G}_2, \epsilon)$ for some radii ϵ , and this justifies the definition of $heavy_{GES_2^*}^{GES_1^*}$. Moreover, weight functions for gestalt composition spaces are not topologically invariant, and therefore, even if Gf^* were a homeomorphism, for which the function $DMet_{GES^2}^{GES^1}$ could then be redefined as $heavy_{GES_2^*}^{GES_1^*}(\overline{G}, \epsilon) := heavy_n^{GES_1^*}(Gg^{-1}(\overline{G}), \epsilon)$, this would not necessarily imply that $heavy_{GES_2^*}^{GES_1^*} = heavy_n^{GES_2^*}$. In other words, the Gg -image of the MET from the domain space does not have to correspond well with the actual MET of the co-domain:

Formalization 32. *The comparison of two motivic analyzes corresponds to ‘measure the non-commutativity’ of the dashed diagram in the bottom right diagram of the MET-comparison diagram for GES_1^{*S} and GES_2^{*S} with respect to a well-defined and continuous gestaltic composition function.*

Finally, we comment on the continuity of g :

Lemma 33. *Given a function $g : (MOT_1^*, \mathcal{T}_{t_1, d_1}^{P_1^*}) \rightarrow (MOT_2^*, \mathcal{T}_{t_2, d_2}^{P_2^*})$ for two motivic composition spaces MOT_1^* and MOT_2^* , the continuity of g is necessary and sufficient for the construction of the MET-comparison diagram for GES^{1*} and GES^{2*} with respect to Gg .*

⁹i.e. k is a bijective map such that for all $x, y \in \mathbb{R}_+ : x < y \Rightarrow k(x) < k(y)$.

In other words, the continuity of g is necessary and sufficient for comparing the Gg -image of MOT_1^* 's MET with the MET of MOT_2^* . Indeed, the continuity of g implies the well-definition and the continuity of the gestaltic composition function Gg . The well-definition of Gg makes possible to compare the gestalt of a motif M (in the left MET) with the gestalt of its image $g(M)$, since $g(Ges_{t_1}^{P_1} \star(M)) = Ges_{t_2}^{P_2} \star(g(M))$. Otherwise, two motives with same gestalt in the MOT_1^* could be mapped to motives with different gestalts in MOT_2^* , making comparison of gestalts problematic. Also the continuity of Gg implies that the small gestalt relation is preserved which is essential since the MET involves the closures of gestalts (see the horizontal lines in Figures 35 and 36).

6 Application to the *Art of Fugue*

6.1 Motivic Space and MET of the 8-Note Main Theme

We exemplify the setup leading to a motivic topology in a way that we can easily recognize the usual American Set Theory (Forte [9], Lewin [12], Morris [21], Rahn [24]) approach to contours that has been extended, in our approach, to topological spaces. See Buteau & Mazzola [6] and Buteau[4] for a more complete description of this example.

We consider the space $\mathbb{R}^{\{O,P,D\}}$ and fix the parameters O, P , and D in a way which is standard in Mathematical Music Theory (Mazzola [13]): for the pitch values, we select the usual gauge with $C_4 = 0$, and the chromatic pitch set being parameterized by the integers, i.e. $C\sharp_4 = D\flat_4 = 1, D_4 = 2$, etc. Duration values are taken by the prescription that 1 in the O -coordinate corresponds to the literal mathematical value of 4/4 duration. The first tone of a score is given onset value 0.



Figure 34.

Main theme of Bach's Kunst der Fuge, 8-tone and 12-tone version.

We first suppose that our musical composition S contains only the eight notes from Bach's *Art of Fugue* as shown in Figure 34 top bar. We have the set of the score's notes:

$$S = \left\{ \begin{array}{l} (0, 2, \frac{1}{2}), (\frac{1}{2}, 9, \frac{1}{2}), (1, 5, \frac{1}{2}), (\frac{3}{2}, 2, \frac{1}{2}), \\ (2, 1, \frac{1}{2}), (\frac{5}{2}, 2, \frac{1}{4}), (\frac{11}{4}, 4, \frac{1}{4}), (3, 5, \frac{1}{2}) \end{array} \right\}$$

and select the collection $MOT^*(S)$ of motives for the musical score S as containing all motives with cardinality between 2 and 8. Therefore, the collection $MOT(S)$ contains

$\binom{8}{2} = 28$ motives of cardinality 2, $\binom{8}{3} = 56$ of cardinality 3, $\binom{8}{4} = 70$ of cardinality 4, $\binom{8}{5} = 56$ of cardinality 5, $\binom{8}{6} = 28$ of cardinality 6, $\binom{8}{7} = 8$ of cardinality 7, and $\binom{8}{8} = 1$ of cardinality 8, which makes a total of 247 motives. We choose the COM-Matrix shape type $t = Com$, the paradigmatic group is $P = CP$, and the distance $d_{t,n} = \sqrt{2} \cdot RED_n$.

If we consider the two motives $Motif_1 = \{(\frac{1}{2}, 9, \frac{1}{2}), (\frac{3}{2}, 2, \frac{1}{2}), (\frac{11}{4}, 4, \frac{1}{4})\}$ and $Motif_2 = \{(0, 2, \frac{1}{2}), (2, 1, \frac{1}{2}), (\frac{3}{2}, 2, \frac{1}{2})\}$ from Figure 1. Their Com -abstract images of $Motif_1$ and $Motif_2$ are $Com_3(Motif_1) = (-1, -1, 1)$ and $Com_3(Motif_2) = (0, -1, -1)$. The gestalt for the counterpoint paradigmatic group CP of $Motif_1$ is the collection of all motives in $MOT_3^*(S)$ such that their images through the mapping Com is one of the following four abstract motives: $(-1, -1, 1)$, $(1, 1, -1)$, $(-1, 1, 1)$, or $(1, -1, -1)$ (corresponding respectively to the abstract motif $Com_3(Motif_1)$, its inversion, its retrograde, and its inversion composed with the retrograde). We get the following number of gestalts (motif classes): there are 2 gestalts of motif cardinality 2, 5 of cardinality 3, 18 of cardinality 4, 34 of cardinality 5, 25 of cardinality 6, 8 of cardinality 7, and 1 of cardinality 8.

The distance ('CSIM value') between the two motives $Motif_1$ and $Motif_2$ is then $d_{Com}(Motif_1, Motif_2) = \sqrt{2} \frac{((-1-0)^2 + (-1--1)^2 + (1--1)^2)^{1/2}}{3} = \sqrt{2} \cdot \frac{\sqrt{5}}{3}$ and the distance between their gestalts is $Gd_{Com}^{CP}(Ges_t^P(Motif_1), Ges_t^P(Motif_2)) = \sqrt{2} \cdot \min\{\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}, \frac{3}{3}, \frac{1}{3}\} = \frac{\sqrt{2}}{3}$. The minimal gestalt distance between any two motives (with same cardinality) of different gestalts is 0.202, and the maximal one is 1.732. Finally, given an $\epsilon > 0$, the ϵ -neighborhood of $Motif_1$ is the set of all motives M in $MOT(S)$ of cardinality 3 to 8 such that:

$$\text{a) } \min \left\{ \begin{array}{ll} d_3(Com(M), (-1, -1, 1)) & , d_3(Com(M), (1, 1, -1)), \\ d_3(Com(M), (-1, 1, 1)) & , d_3(Com(M), (1, -1, -1)) \end{array} \right\} < \epsilon,$$

if $card(M) = 3$;

$$\text{b) } \min \left\{ \begin{array}{ll} d_3(Com(M'), (-1, -1, 1)) & , d_3(Com(M'), (1, 1, -1)), \\ d_3(Com(M'), (-1, 1, 1)) & , d_3(Com(M'), (1, -1, -1)) \end{array} \right\} < \epsilon,$$

for a suitable submotif M' of cardinality 3, of M with $card(M) = 4, 5, 6, 7$ or 8.

By definition of ϵ -neighborhoods, motives with cardinality 2 cannot be in the neighborhood of $Motif_1$. This completes the setup of the motivic composition space $MOT^*(S)$ of the score S . We then evaluate the function *Weight* but because of calculation constraints, it is evaluated at selected radii $r = 0.2, 0.25, \dots, 1.65$. We fix the tolerance to $tol = 2$, and evaluate *heavy*₂.

Finally, we represent intuitively the motivic evolution tree—MET— (Figure 35) of this topological structure by the following: we construct a coordinate system for which in the vertical axis we consider the radius variable ('similarity variable')—growing from top to bottom— and in the horizontal axis the gestalt cardinality variable. For radius $\epsilon > 0$, we graphically represent the shapes of gestalts G with $(G, \epsilon) \in \Theta$ at coordinate $(card_t(G), \epsilon)$. These gestalts G are ordered by their *heavy*_n images, and this order is represented by the gray intensity (value 1: black; value 0: white) of their shape representation. Then we link gestalts which are small gestalts from each other. Since

this relation is transitive we link only gestalts with consecutive cardinalities. We don't repeat gestalts at consecutive radii. Dashed boxes around a gestalt means that the latter has already appeared in the MET at smaller neighborhood radius. Flags at gestalts correspond to their 'identity' numbers (e.g. "3-13" stands for the 13th gestalt in the lexicographically ordered list of gestalts with cardinality 3) and their multiplicities (e.g. "M:12" stands for $mult = 12$).

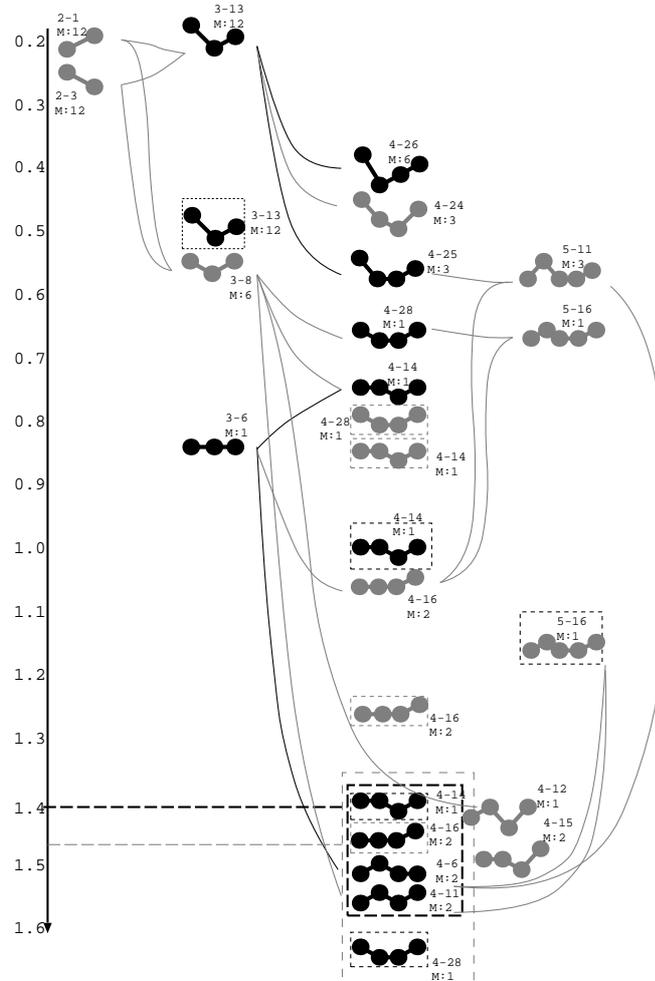


Figure 35.

This graphic shows the motivic evolution tree (MET) at tolerance 2, of the 8-tone main theme of Bach's Art of Fugue. As a global information from this tree, we should understand: when looking from top to bottom, i.e. when the neighborhood radius (similarity threshold) is growing, we view (in black) the evolution of the score's germinal motif gestalt obtained from our motivic analysis.

6.2 Mapping of the Art of Fugue's METs

We want to compare Bach's *Art of Fugue* 8-tone and 12-tone main themes (see Figure 34) together in order to address the still debated question concerning the length of the main theme: *is the theme the 8 or the 12 first notes?*

We therefore construct MET-comparison diagrams: we denote the 8-note theme $kf8$ and the 12-note theme $kf12$, and consider the collection $MOT^*(kf8)$ of all 2-to-8 note motives in $kf8$, as in the previous example, as well as $MOT^*(kf12)$ of all 2-to-8 note motives in $kf12$ (with a total of 3784 motives). We then have the following four gestalt composition spaces

$$\begin{aligned} KF8_{Id} &:= (GES^*(MOT^*(kf8)), \mathcal{GT}_{Com\ REd}^{Id}), \\ KF8_{CP} &:= (GES^*(MOT^*(kf8)), \mathcal{GT}_{Com\ REd}^{CP}), \\ KF12_{Id} &:= (GES^*(MOT^*(kf12)), \mathcal{GT}_{Com\ REd}^{Id}), \\ KF12_{CP} &:= (GES^*(MOT^*(kf12)), \mathcal{GT}_{Com\ REd}^{CP}) \end{aligned}$$

with respectively 118, 93, 1675, and 1372 gestalts¹⁰. Moreover, in the case of $kf8$, the minimal gestalt distance between two different gestalts is 0.202 for both paradigms, and the maximal one is 1.732 for Id and 1.22 for CP . For the composition $kf12$, the minimal gestalt distance between two different gestalts is 0.177 for both paradigms, and the maximal one is 1.79 for Id and 1.27 for CP . Since $MOT^*(kf8) \subset MOT^*(kf12)$ and $Id \subset CP$, the topological spaces can be linked as in the following diagram

$$\begin{array}{ccc} (GES, \mathcal{GT}_{Com\ REd}^{Id}) & & \\ \downarrow & \swarrow \text{dashed} & \\ (GES, \mathcal{GT}_{Com\ REd}^{CP}) & & \\ & \nwarrow \text{dashed} & \\ & & \end{array} \quad \begin{array}{ccc} & & \\ & \xrightarrow{KF8_{Id}} & \\ & \downarrow & \\ & \xrightarrow{KF8_{CP}} & \\ & \downarrow & \\ & \xrightarrow{KF12_{Id}} & \\ & \downarrow & \\ & \xrightarrow{KF12_{CP}} & \end{array}$$

in which all plain arrows in the right square are their respective gestaltic composition set-identity function, the very left down arrow the gestaltic set-identity functions between two gestalt motivic spaces (infinite set of all gestalts), and the dashed arrows are the natural injective mappings (finite relative spaces). The diagram commutes, and by Corollary 31, these four gestaltic composition set-identity functions and the gestaltic set-identity function are continuous.

Therefore, by Lemma 33, we have respectively four MET-comparison diagrams. See Figure 36 for a representation of the MET-comparison diagram corresponding to $Gid^* : KF8_{CP} \rightarrow KF12_{CP}$. In this figure, colored em-boxed gestalts G in the left MET means that $\overline{G} \in \Theta_{KF8_{CP}} \cap \Theta_{KF12_{CP}}$. The colored lines between a gestalt G in the left MET and a gestalt H in the right MET represents the small gestalt relation between gestalts $Gid^*(G)$ and H : $Gid^*{}^S(\overline{G}) \subset \overline{H} \in \Theta_{KF12_{CP}}$ where $\overline{G} \in \Theta_{KF8_{CP}}$. This is a consequence

¹⁰For details on each of these spaces, see Buteau [4, Section 4.4.2]

of the continuity of *Gid**.

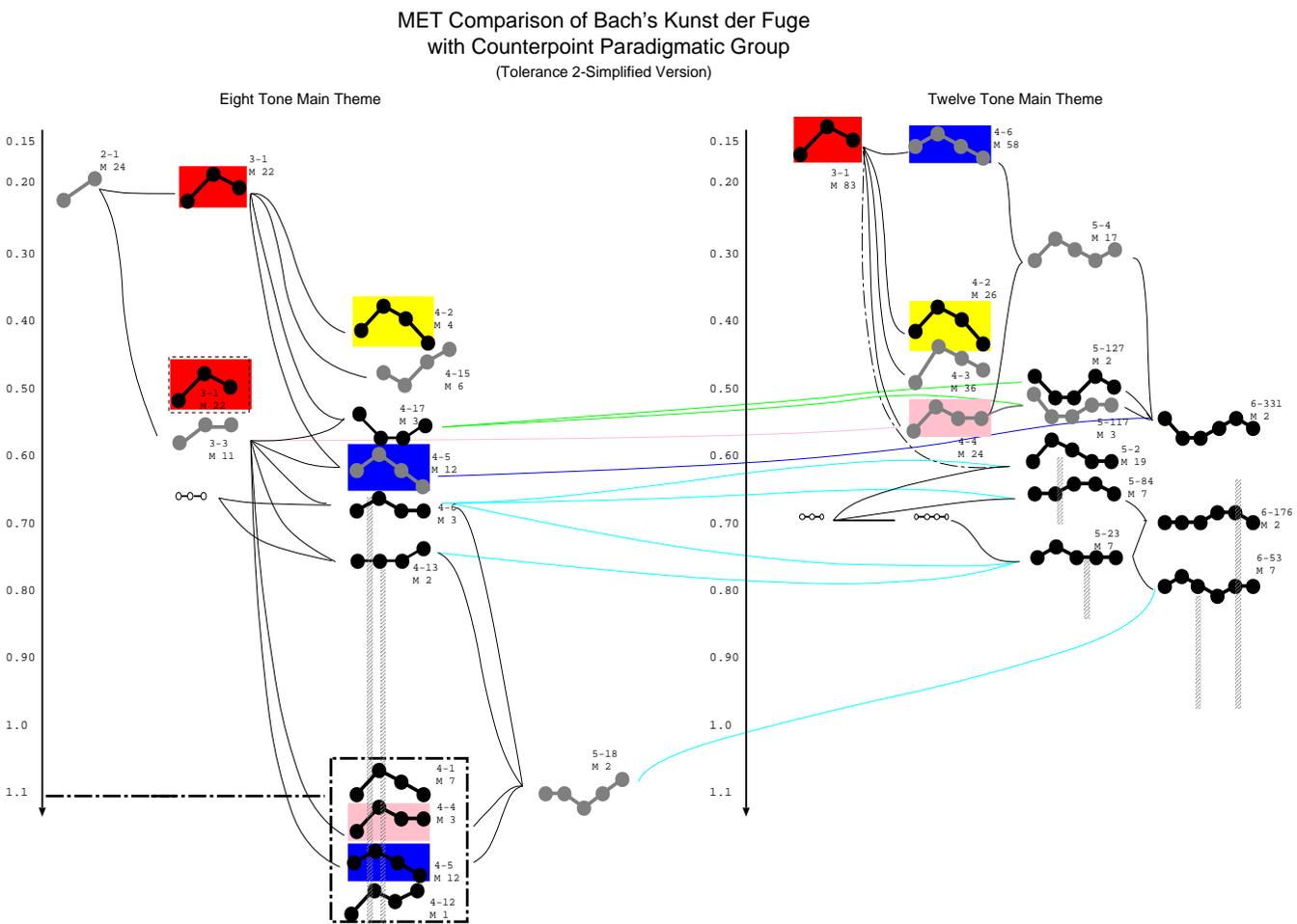


Figure 36.

To the left, we show the tree $MET(8, CP)$ and to the right, the tree $MET(12, CP)$. Compared to the representation in Figure 35, this one is slightly simplified: the vertical lines show the radius interval where the specific gestalt remains the heaviest or the second heaviest. The horizontal connections from $MET(8, CP)$ to $MET(12, CP)$ indicate the gestalt inclusion relation.

We restrict here to the comparison of the trees for the counterpoint paradigm¹¹, i.e., we compare $MET(8, CP)$ and $MET(12, CP)$ whereas e.g. $MET(8, CP)$ (left in Figure 36) refers to the gestalt motivic space $(GES^*(MOT^*(k.f8)), \mathcal{GT}_{Com, REd}^{CP})$ of all motives of the 8-tone themes. When we look at the leading gestalts in $MET(12, CP)$ (right in Figure 36), some of them are already present in $MET(8, CP)$. This is the case for gestalts 3-1, 4-2, 4-4, and 4-6, as shown in colored boxes in the METs. The others, i.e. 5-2, 5-4, 5-23, 5-84, 5-127, 6-53, 6-176, and 6-331 are all gestalt extensions of most of the remaining heaviest gestalts, i.e. 4-6, 4-13, 4-17, which live in the subspace of the 8-tone theme. Topologically, this means that the extended heaviest gestalts are in every neighborhood of their respectively included (heaviest) subspace gestalts. We have similar statements (Buteau [4]) for the 3 other MET-comparison diagrams.

Summarizing, either a leading gestalt of the 12-tone theme is already found in the MET of the 8-tone theme, or it is contained in every neighborhood of a corresponding heaviest gestalt of the 8-tone theme. However, all of the extended gestalts of the 12-tone theme, except 5-2, involve tones outside the 8-tone theme. In other words, the added tones do support the motivic ‘substance’, but topologically, this substance is ‘arbitrary near’ to heaviest gestalts of the 8-tone theme, and we concluded (Buteau & Mazzola [6]):

Result 37. (Buteau & Mazzola [6]) *The significant¹² motives of Bach’s Kunst der Fuge’s 8-tone main theme are part of the significant motives of the 12-tone theme, but the last four notes do not generate a proper extension to the set of significant motives. However, the last four notes are all related to the significant motives of the 12-tone theme. In other words, the extension to twelve tones is ‘substantial’, but it is not a proper extension.*

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¹¹We observe that the choice of the counterpoint paradigm for our analysis of a Bach composition does support indeed *traditional* music analysis.

¹²Observe that our concept of a ‘significant motif’ is a mathematical concept. This means that this result may confirm musicological intuition but it is a completely rational fact, which significantly transcends prescientific knowledge.

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