RUBATO's MeloTopRUBETTE for Topological Analysis of Melodic Paradigms

by

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Abstract

In the framework of the Distributed RUBATO platform, we propose an improved version MeloTopRUBETTE of RUBATO's successful module MeloRUBETTE for motivic analysis. It fits with computational steering interaction and implements topological and sheaf-theoretical aspects of motive theory, such as motivic evolution trees (MET) and stalk dimensions for weight function sheaves. We present the theory and the algorithm flow chart of our model of motivic analysis following Rudolph Reti's approach.

Key Words: Motive Analysis, Motivic Topologies, Mathematical Music Theory, Melo(Top)RUBETTE module, RUBATO software, Quantification of Topologies, Function Spaces, Weight Functions.

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1 Introduction

We propose an improved version of the module MeloRUBETTE of the software platform RUBATO for musical analysis and performance [9], [15]. This module of the now developed Distributed RUBATO platform [11] (chapter 40) is called MeloTopRUBETTE[1] and implements our mathematical motive theory. Its design follows the requirements of computational steering, i.e. with interactive control of the computational process during its execution

The motivic analysis of a score has always been a main subject in musicology, but the very complexity of the motivic concepts and configurations made it impossible to 'traditional' musicologists to transcend pure intuition. Nonetheless, crucial ideas about motivic analysis were proposed by Rudolph Reti and Michael Kopfermann [13]. In order to find out which motives are the germs and motors in the evolution of the motivic content, they suggested an immanent approach: One should not impose the germinal motives from outside, but construct the germs from a thorough analysis of all possible motif structures and relations within a given composition. So musicology is facing two main problems: To formalize the concept framework dealing with motives, and to establish a formally valid model of motivic germs.

We propose a solution to these two problems: In [5], we introduced a model which formally conceptualizes Reti's and Kopfermann's approach. The present operationalization of our motivic theory implements modular algorithms, making possible to modify the model during its empirical testing process.

Let us briefly recall our mathematical model on motivic analysis of music. In [8], a mathematical theory of musical structures was developed which is based on local charts and global discrete varieties of tones in specific parameter spaces. In this approach, a motif is a local chart of a specific type. A motivic interpretation of a musical composition is a global variety, i.e. an interpretation of the score by a selected covering system of motives. Topological spaces built on motivic interpretations are called motivic spaces. They are defined from orbits of motives under actions of transformation groups, from metrical similarity between motives with same cardinality, and from submotif relations. It is important to observe that this topological structure is defined on the set of all motives (with different cardinalities) of a score. The motivic space of a score is however non-intuitive (not Haussdorff, only

[1] The name stands for: RUBETTE of Melodic Topologies

of type T_0). For this reason, a more geometric perspective has been realized in the original MeloRUBETTE by giving each motive, and thereby each tone of the score, a weight that corresponds to their topological "presence" and "content", see chapter 22.9 in [11]. Corresponding analytical results have been used to investigate and produce performances of classical compositions [1], [2], [3], [4], [10], [14].

The problem of geometrization of motivic spaces deals with the main problem of motive theory: the exhibition of germinal motives. It has been further investigated and led to the concept of a motivic evolution tree (MET), the graphical representation of an overall spectrum of a score's motivic structure (motivic space) [6]. The first trace of a sheaf-theoretic perspective related to the MET is given in [7]. In fact, the sheaves give rise to coordinate functions (the global sections) which yield embeddings of motivic topologies in real vector spaces. First empirical investigations [6], [7] on small compositional units, such as the main theme of Bach's "Kunst der Fuge", demonstrated the viability of our approach. The present MeloTopRUBETTE opens the path to deeper and more meaningful empirical investigations.

As a major improvement of the MeloTopRUBETTE against the MeloRUBETTE is the interactive control of the ongoing computational process; it makes possible to extend and to improve the model "on the flight". This computational steering approach helps understanding and also modifying the mathematical model on motivic analysis of music. Calculations extend to classes (gestalts) of motives, thereby reducing considerably the amount of calculations. There are also more possibilities of shape types ("types of abstraction") and of paradigmatic transformation groups. In particular, the "diastematic index" shape type from the MeloRUBETTE, for which contour vectors with entries -1, 0, and 1 represent the diastematic movement of a motif, has been refined to yield a motivic space, a topological structure which is impossible to define on the "diastematic index". Moreover, the contour similarity "theory" of the American Set Theory is a special case of our implementation, in which their contour similarity concepts are extended to a topology on the space of all motives of a score, i.e. a structure in which a similarity concept between motives of different cardinalities is introduced.

Finally, a major improvement of the MeloTopRUBETTE is the enrichment of the output: In the MeloRUBETTE, the output consisted of weights on tones of the analyzed score, whereas in the MeloTopRUBETTE, we propose, in addition to the interactive control of the computational process, weights on tones, weights on motives and on gestalts with their graphics, the motivic evolution tree, the dimensions of real vector spaces of the presence, content, or weight functions, dimensions of respective stalks of related function sheaves, and, as useful feature, a motif and gestalt information interactive window which outlines e.g. the motif's notes in the score, its weights at all radii, its gestalt (class of motives), its image in the motivic evolution tree, etc.

2 Motivic Spaces

The formal definition of a motivic space presupposes a motif concept. We want to restrict our attention to a minimal parameter setup in order to make the essential clear, and we refer the reader to the end of this section for a simple example leading to a motivic topology. Consider the space $\mathbb{R}^{\{O,P,D,L,G,C\}} \cong \mathbb{R}^6$ of tone parametrization for which the parameters are respectively *onset, pitch, duration, loudness, glissando* and *crescendo*, and consider also the canonical projection

$$p_O : \mathbb{R}^{\{O,P,D,L,G,C\}} \to \mathbb{R}^{\{O\}}$$

on the axis of onset events. Denote $\mathbb{R}^{OP\cdots} \subseteq \mathbb{R}^{\{O,P,D,L,G,C\}}$ the space of notes parametrized by at least onset and pitch parameters. A *motif* $M = \{m_1, ...m_n\}$ is a non-empty finite subset in $\mathbb{R}^{OP\cdots}$ such that $p_O(M)$ is a bijection.

A motif is therefore a finite set of notes $m_i = (o_i, p_i, ...)$ such that only one note is heard at a given onset. A submotif M' of a motif M is a motif such that $M' \subset M$. The set of motives is denoted by MOT, which is the disjoint union of all subsets MOT_n of motives M of cardinality $card(M) = n$. If we are given a score S, a collection of motives with all notes living in S is denoted by $MOT(S)$, which is the disjoint union of the subsets

$$MOT_n(S) = MOT(S) \cap MOT_n.$$

Motives are always mapped to abstractions, for example for contour information. This means that we have a family $t = (t_n)$ of maps

$$t_n : MOT_n \to \Gamma_{t,n}$$

into mutually disjoint[2] sets $\Gamma_{t,n}$ of *abstract motives* of *abstract cardinality* n. The family t is called the *shape type*, whereas the elements of $\Gamma_{t,n}$ are called

[2]In the general theory, disjointness is not mandatory, however

abstract motives of abstract cardinality n. A typical such map is the contour type $t = Cont$ (which corresponds in the American Set Theory to the *COM matrix*). We have

$$\Gamma_{Cont,n} = \mathbb{Z}^{n(n-1)/2},$$

and if $M = \{m_1, m_2, ..., m_n\} \in MOT_n$ with notes $m_i = (o_i, p_i, ...)$, we set $t_n(M) = (\Delta_{ij})_{1=i<j=n}$, with $\Delta_{ij} = 1$ if the pitch difference $(p_j - p_i)$ of notes m_j and m_i in M is positiv, 0 if the difference is null, and -1 if it is negativ; see [7] for further examples. On each space $\Gamma_{t,n}$ of abstract motives of abstract cardinality n, we suppose also given a pseudo-metric $d_n(x_1, x_2)$, for example the Euclidean metric on $\Gamma_{Cont,n}$. Call the family $d = (d_n)_{n\in\mathbb{N}}$ a pseudo-metric on the shape type t. This induces a pseudo-metric (family) $d_t = (d_{t,n})_{n\in\mathbb{N}}$, which on each MOT_n is defined by $d_{t,n}(M_1, M_2) = d_n(t(M_1), t(M_2))$.

Finally, we suppose that there is a pair of t_n-equivariant group actions $P\times MOT_n \to MOT_n$, $P \times \Gamma_{t,n} \to \Gamma_{t,n}$ for each n. Following Nattiez and Ruwet [12], the group P is called the *paradigmatic group*. The typical example is the pointwise action of the affine counterpoint group CP, which (1) on the space $\mathbb{R}^{OP...}$ of notes acts as the group of affine transformations generated by all the translations, the horizontal reflexions U_p at pitch p (the inversions), and the vertical refexions K_o at onset o (the retrogrades); and (2), on $\mathbb{Z}^{n(n-1)/2}$ is the canonically induced action (translations act trivially, inversions by sign inversion, and retrogrades by sign inversion and index exchange). One also supposes that P acts as a group of isometries[3] (preserving metrical distances) on each $\Gamma_{t,n}$, which clearly induces an action by isometries on MOT_n. We call the inverse image $t^{-1}(P\cdot t_n(M))$ of a P-orbit of a motive's abstract motif $t_n(M)$ its *gestalt*:

$$Ges(M) = t^{-1}(P \cdot t_n(M)).$$

The gestalts define a partition of the total space MOT, the set of gestalts is denoted by GES. It is evidently the disjoint union of the classes GES_n in MOT_n. Its trace in $MOT(S)$ is denoted by $GES(S)$, and the projections are denoted by $\gamma : MOT \to GES$, and $\gamma(S) : MOT(S) \to GES(S)$. We have a (family of) pseudo-metric(s) $Gd_t^P = (Gd_{t,n}^P)_{n\in\mathbb{N}}$ with $Gd_{t,n}^P$ on GES_n, which is defined by

$$Gd_{t,n}^P(Ges(M_1), Ges(M_2)) = \inf_{p\in P} d_{t,n}(p \cdot M_1, M_2).$$

[3]Note that this hypothesis is natural: the distance between two motives should be the same as when one e.g. equally translates both of them!

We now are ready to set forth the topological framework. For a given data set as described above, suppose that $\varepsilon > 0$ is a real number, and that $M \in MOT_n$. Then we set

$$U_\varepsilon(M) = \{N \mid \exists N' \in MOT_n,\ N' \subset N\ :\ \inf_{p \in P} d_{t,n}(p \cdot M, N') < \varepsilon\},$$

and call it the *ε-neighborhood of M*. Observe that it is at this point that we link motives of different(!) cardinalities. If our setup fulfills the inheritance property [5], [7], the system of ε-neighborhoods defines a base for a topology on MOT. For example, the contour type together with the Euclidean metric fulfills the inheritance property. The space MOT with this topology is called the *motivic space*, its relativization to $MOT(S)$ is called the *motivic space on S*.

We now introduce a topology on GES. To this end, we need a relation on gestalts corresponding to the *submotif relation* on motives: If G and G' are two gestalts, we say that G' is a *small gestalt of* G, in signs: $G' \prec G$, iff there are motives $M \in G$, $M' \in G'$ such that $M' \subset M$. Then the sets

$$U_\varepsilon(G) = \{H \mid \exists H' \in GES_n,\ H' \prec H\ :\ Gd^P_{t,n}(G, H') < \varepsilon\}$$

form a topological base of GES. With this topology (the structures leading to this topology be implicitely assumed), GES is called a *motivic gestalt space*, for a score S the relative space $GES(S)$ is called a *motivic gestalt space on S*. We denote the *ε-neighborhood of a gestalt* G in $GES(S)$ by

$$SU_\varepsilon(G) := U_\varepsilon(G) \cap GES(S),$$

and denote $mult(G)$, called the *multiplicity of* G, the cardinality of all motives in $MOT(S)$ in gestalt G.

Theorem [5]. If the pseudo-metric Gd^P_t is a metric (i.e., if this is the case for all n), then the canonical maps $\gamma : MOT \to GES$, and $\gamma(S) : MOT(S) \to GES(S)$ are open continuous maps onto the quotients, and the topologies on GES and $GES(S)$ are the quotient topologies.

Example. *We exemplify the setup leading to a motivic topology. See [6] for a more complete description. Consider the space $\mathbb{R}^{\{O,P,D\}}$. We fix the parameters O, P, and D in a way which is standard in Mathematical Music Theory [8]: For the pitch values, we select the usual gauging with $C_4 = 0$,* *and the chromatic pitch set being parametrized by the integers, i.e.* $C\sharp_4 = D\flat_4 = 1$, $D_4 = 2$, *etc. Duration values are taken by the prescription that* 1 *in the O-coordinate corresponds to the literal mathematical value of* $4/4$ *duration. The first tone of a score is given onset value 0.*

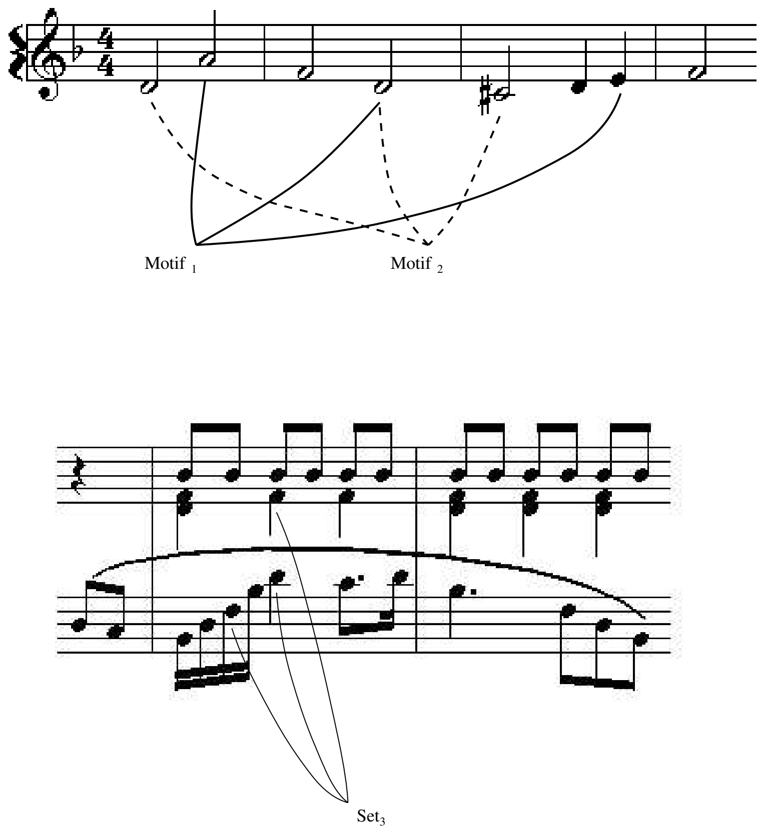

Figure 1. Examples of sets of notes which form a motif: $Motif_1$ and $Motif_2$; and which do not form a motif: Set_3.

Consider the Figure 1. *The two sets* $Motif_1 = \{(1/2, 9, 1/2), (3/2, 2, 1/2), (11/4, 4, 1/4)\}$ *and* $Motif_2 = \{(0, 2, 1/2), (2, 0, 1/2), (3/2, 2, 1/2)\}$ *are motives but the set* Set_3 *is clearly not a motif.*

Suppose that our score S *contains only the above eight notes from Bach's Kunst der Fuge 8-tone Main Them. An example of collection* $MOT(S)$ *of motives for the score* S *is all motives with cardinality between 2 and 4.* $MOT(S)$ *contains* $\binom{8}{2} = 28$ *motives with cardinality 2,* $\binom{8}{3} = 56$ *with cardinality 3, and* $\binom{8}{4} = 70$ *with cardinality 4, which makes a total of 154 motives.*

The abstract images with respect to the shape type $Cont$ *of the two motives* $Motif_1$ *and* $Motif_2$ *are* $Cont_3(Motif_1) = (-1, -1, 1)$ *and* $Cont_3(Motif_2) = (0, -1, -1)$. *The gestalt for the counterpoint paradigmatic group* CP *of* $Motif_1$ *is the collection of all motives in* $MOT_3(S)$ *such that their images through the mapping* $Cont$ *is one of the following four abstract motives:* $(-1, -1, 1)$, $(1, 1, -1)$, $(-1, 1, 1)$, *or* $(1, -1, -1)$ *(corresponding respectively to the abstract motif* $Cont_3(Motif_1)$*, its inversion, its retrograd, and its inversion composed with the retrograde). Using the MeloTopRUBETTE for identifying the motives together, we get the following number of gestalts (classes of motives): there are 3 gestalts with motif cardinality 2, 13 with cardinality 3, and 30 with cardinality 4.*

Now if we consider the Euclidean distance, then the distance between the two motives $Motif_1$ *and* $Motif_2$ *is*

$$d_{Cont}(Motif_1, Motif_2) = \left((-1-0)^2 + (-1--1)^2 + (1--1)^2\right)^{1/2} = \sqrt{5}.$$

The distance between their respective gestalts is

$$Gd^{CP}_{Cont}(Ges(Motif_1), Ges(Motif_2)) = min\{\sqrt{5}, \sqrt{5}, 3, 1\} = 1.$$

Finally the ε*-neighborhood of* $Motif_1$ *is the set of all motives* M *in* $MOT(S)$ *with cardinality 3 or 4 such that:*

1. *If* $card(M) = 3$*, then*

$$\min\{ \; d_3(Cont(M), (-1,-1,1)), d_3(Cont(M), (1,1,-1)), \; d_3(Cont(M), (-1,1,1)), d_3(Cont(M), (1,-1,-1)) \; \} < \varepsilon;$$

2. *If* $Card(M) = 4$*, then there is a submotif* $M' \subset M$ *with cardinality 3 such that*

$$\min\{ \; d_3(Cont(M'), (-1,-1,1)), d_3(Cont(M'), (1,1,-1)), \; d_3(Cont(M'), (-1,1,1)), d_3(Cont(M'), (1,-1,-1)) \; \} < \varepsilon.$$

By definition of ε*-neighborhoods, motives with cardinality* 2 *cannot be in the neighborhood of* $Motif_1$.

This completes the setup of the motivic space $MOT(S)$ *of the score* S.

3 Presence, Content, and Weight Functions

Motivic gestalt spaces are [5] of type T_0, 'almost' of type T_1, and, if $MOT(S)$ contains motives with different cardinalities, not at all of type T_2 (Hausdorff), which excludes any intuitive representation of the topological structure. Therefore, in order to provide us with a more geometric picture of the motivic spaces and motivic gestalt spaces on a score S, we introduce real-valued functions which account for the topological relations on these spaces. Observe that we have to take into account the intrinsic neighborhood asymmetry between gestalts.

The 'presence' of a gestalt is the magnitude of its neighborhood and its 'content' is the frequency of its appearance in other motives' neighborhoods: First consider two gestalts G and H in $GES(S)$ with $G \in GES_n$, and a neighborhood radius $\varepsilon > 0$. If $H \in SU_\varepsilon(G)$, then one measures the presence of gestalt G in gestalt H (or, inversed roles: H being contained in G) by the intensity integer

$$Int_\varepsilon(H|G) = card\{H' \prec H \mid H' \in GES_n \cap SU_\varepsilon(G)\} \cdot mult(H)$$

Since the higher cardinality difference between G and H the higher the probability that $Int_\varepsilon(H|G) \neq 0$, we weight the intensity by $1/2^{(card(H)-card(G))}$. The *presence and the content of* G *at radius* $\varepsilon > 0$ is defined [4] as

$$Presence_\varepsilon(G) := \sum_{H \in GES(S)} 1/2^{(card(H)-card(G))} \cdot Int_\varepsilon(H|G)$$

$$Content_\varepsilon(G) := \sum_{H \in GES(S)} 1/2^{(card(G)-card(H))} \cdot Int_\varepsilon(G|H)$$

and the *weight of gestalt* G *at radius* ε is

$$Weight_\varepsilon := Presence_\varepsilon(G) \cdot Content_\varepsilon(G)$$

Note that these *quantification functions*, presence, content and weight functions, on gestalts can be extended to motives.

The information related to S, which is set forth by the quantification functions, is articulated in sheaves of function vector spaces F on $GES(S)$,

[4]There are more parameters in the general definition [7] of these two functions.

whose sections $F(U)$ are linear combinations of determined systems of presence functions, content functions, or weight functions, respectively: Consider, for a given $\varepsilon > 0$, a quantification function f_ε. We define a presheaf F' through

$$F'(GES(S)) = \mathbb{R} < f_\varepsilon, \varepsilon >$$

and for any $\varepsilon' > 0$ and gestalt G in $GES(S)$

$$F'(U_{\varepsilon'}(G)) = F'(GES(S)|U_{\varepsilon'}(G)).$$

The sheafification F of F' for presence, content, and weight, respectively, constitutes a system of local coordinate functions on motivic gestalt spaces.

4 The Flow Chart for the Calculations in the MeloTopRUBETTE

The topological and sheaf-theoretic information for a score S is implemented in the MeloTopRUBETTE according to the flow chart shown in Figure 2. Each pink box of this flow chart is provided with an output denotator for visual and/or sonic representation on the RUBATO's PrimaVista Browser [11] (chapter 40).

There is a double input. On one side, input (00) contains the score S as a denoTeX or MIDI file. On the other side, input (01) contains 20 input parameters: the score's collection of motives parameters (e.g. minimal and maximal motif cardinality), the topological parameters (e.g. shape type and paradigmatic group), the similarity radii set parameters (in order to obtain relevant radii for evaluating the quantification functions), and the optional output parameters (e.g. with or without motivic evolution tree).

In (1) the score is simplified into a collection of notes. Observe that any other events such as pauses are not present in this simplified *melodic score*. The collection of all motives of the score to be analysed is created in (2). The abstract images (shapes) of the motives is calculated in (3). The important step (4) corresponds to the identification of motives (from (1)) into gestalts (classes of motives) with respect to an action of a paradigmatic group. In (5) the small gestalt relation between all pairs of gestalts (from (4)) is evaluated. Note that this relation on gestalts is the corresponding submotif relation on motives, and it is exactly after this step that all other calculations are done on gestalts instead of on motives, until we finally come back, at (22), to the level of motives.

Directly resulting from step (5) the collection of generic gestalts, i.e. the maximal elements if the motivic gestalt space $GES(S)$ under the topological dominance relation, is created in (6), and the N_0-matrix in (7). The N_0-matrix of small gestalt relations corresponds to the ε_0-neighborhoods, $\varepsilon_0 \to 0$, of all gestalts.

The distance function between all pairs of gestalts with same cardinality is evaluated in (8), as well as the collection of all changing distances, i.e., those radii of neighborhoods, where neighborhoods (they are all finite and only differ for specific "jumping" radii) become larger while increasing the radii. Box (9) is the unit where the collection of all neighborhood radii at which quantification functions, i.e. presence, content and weight functions, will be evaluated, is created.

The boxes from (10) to (14), and from (24) to (26) comprise a cycle for the calculation of stalk dimensions of function spaces for content and presence. The corresponding output is obtained in the box (15). The running variable "j" deals with the enumeration index of gestalts in S. The output (18) and (19) comprises dimensions of stalks and global section spaces. Box (21) delivers the motivic evolution tree of S. Finally, weights of motives and of notes of S are calculated respectively in (22) and (23).

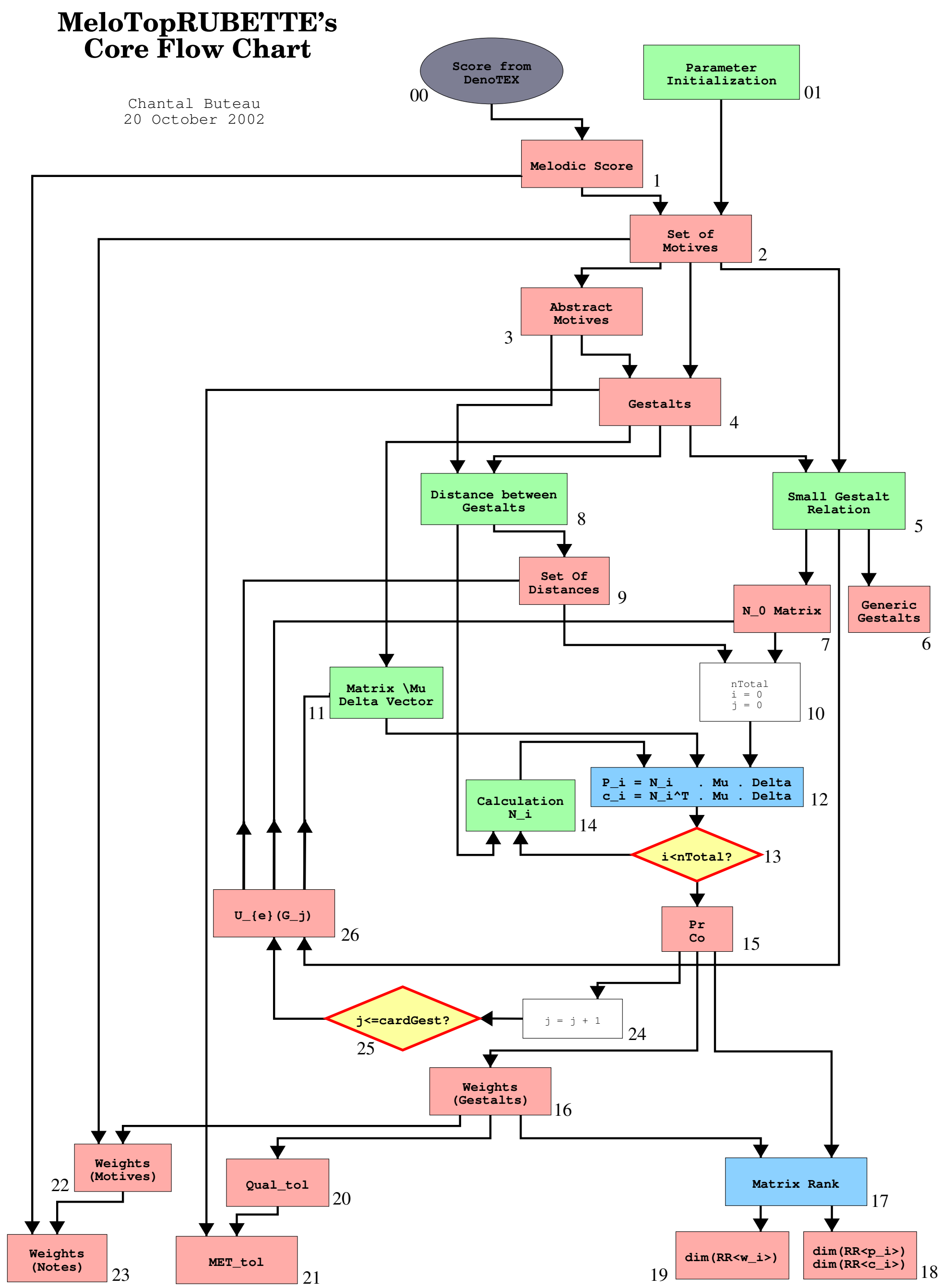

Figure 2. The flow chart of the MeloTopRUBETTE.

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