BROCK UNIVERSITY MATHEMATICS MODULES

11A2.8: The Quadratic Formula

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WWW

- What it is: The quadratic formula can be used to solve all quadratic equations.
- Why you need it: The quadratic formula is useful when the zeros of a quadratic function cannot be easily determined using simple factoring.
- When to use it: Use the quadratic formula when you need to solve a quadratic equation, or determine the zeros of a quadratic function, and it's difficult to use factoring.

PREREQUISITES

Before you tackle this module, make sure you have completed these modules:

11A1 Functions, 11A2.1 Zeros of a Quadratic Function, 11A2.2 Factoring Quadratic Expressions, 11A2.3 Completing the Square, and 11A2.4 Maximum or Minimum Values for Quadratic Functions

WARMUP

Before you tackle this module, make sure you can solve the following exercises. If you have difficulties, please review the appropriate prerequisite modules.

(Answers below.)

1. Factor each quadratic expression.

   (a) $x^2 + 2x + 1$ (b) $x^2 - 5x + 6$ (c) $2x^2 + 3x + 1$

2. Determine the zeros of each quadratic function.

   (a) $y = x^2 - 9$ (b) $y = x^2 - 3x$ (c) $y = x^2 - 16$

3. Complete the square for each quadratic expression.

   (a) $x^2 + 2x + 4$ (b) $x^2 + 6x - 7$ (c) $4x^2 - 2x - 5$

4. Solve each quadratic equation.

   (a) $0 = x^2 + x - 6$ (b) $3 = x^2 + 2x$ (c) $x^2 + 2x + 3 = 0$

Answers: 1.(a) $(x+1)(x+1)$ (b) $(x-2)(x-3)$ (c) $(2x+1)(x+1)$ 2.(a) $x = 3$ and $x = -3$ (b) $x = 0$ and $x = 3$ (c) $x = 4$ and $x = -4$ 3.(a) $(x+1)^2 + 3$ (b) $(x+3)^2 - 16$ (c) $4(x - 1/4)^2 - 21/4$ 4.(a) $x = 2$ and $x = -3$ (b) $x = 1$ and $x = -3$ (c) No real solutions.

Introduction

In previous modules we looked at various ways to find the zeros of quadratic functions, including factoring and completing the square. In this module we'll study the quadratic formula, which allows us to solve any quadratic equation, including ones for which factoring is not easy. The quadratic formula will also allow us to quickly determine that there are no real solutions to a quadratic equation, when that is the case.

FOCUS QUESTION

To help you understand an important aspect of this lesson, focus your attention on this question, which will be answered towards the end of the lesson.

Robin Hood is in a "longest arrow in the air" competition. If Robin's arrow has an initial upward velocity of 50 m/s and reaches a maximum height of 129 m, will he beat Will Scarlet, whose arrow was in the air for 5 seconds, assuming no air resistance? The acceleration due to gravity is 9.8 m/s^2 downward.

One way of understanding what the quadratic formula is and how to use it, is to see where it comes from. In previous modules you have learned how to solve a quadratic equation by completing the square, and this is where we will start in our derivation of the quadratic formula.

Consider a general quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

Now let's solve the equation by completing the square:

$$\begin{aligned}
a\left[x^2 + \frac{b}{a}x\right] + c &= 0\\
a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c &= 0\\
a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c &= 0\\
a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c &= 0\\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c\\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a}\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}\\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a}\\
x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Recall that not every quadratic equation has solutions. Some have two distinct real solutions, some have just one real solution, and some have no real solutions. What the previous development teaches us is that if a quadratic equation does have solutions, then it is always possible to determine them by completing the square. However, some people prefer to just remember the quadratic formula, which is the result of completing the square for a general quadratic equation.

KEY IDEA

The Quadratic Formula The real solutions of the quadratic equation $ax^2 + bx + c = 0$, if there are any, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are no real solutions if and only if $b^2 - 4ac < 0$.

Let's see how to use the quadratic formula in an example. We'll begin with an example of a quadratic equation that can be solved by factoring, so that we'll have an independent way to confirm the results of the formula

EXAMPLE 1

Solve the quadratic equation $0 = x^2 - 3x + 2 = 0$.

SOLUTION

First note that we can factor the quadratic expression to obtain $(x-1)(x-2) = 0$, which has the solutions $x = 1$ and $x = 2$.

Now let's solve the eequation using the quadratic formula and verify that we obtain the same results. To apply the quadratic formula, note that for our example, $a = 1$, $b = -3$, and $c = 2$. Carefully note that the sign of the coefficient is included in the value, so that $b = -3$; a common mistake is to forget about the sign and *incorrectly* use 3 as the value for b.

With the stated values for a, b, and c, we can write the solutions of the quadratic equation as

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} \\
&= \frac{3 \pm \sqrt{9 - 8}}{2} \\
&= \frac{3 \pm \sqrt{1}}{2} \\
&= \frac{3 \pm 1}{2} \\
&= \frac{2}{2} \qquad \text{and also} \qquad x = \frac{4}{2} \\
x &= 1 \qquad \text{and also} \qquad x = 2
\end{aligned}$$

Thus, the quadratic formula leads to the same results as factoring: the two solutions to the equation are $x = 1$ and $x = 2$.

EXAMPLE 2

Solve the quadratic equation $0 = x^2 - 9$

SOLUTION

Recall that this is one of the warmup questions, and you can solve it by factoring, but let's make sure the quadratic formula gives the same results. First observe that $a = 1$, $b = 0$ (because there is no "x" term), and $c = -9$. The solutions are:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-0 \pm \sqrt{0^2 - 4(1)(-9)}}{2(1)} \\ &= \pm\frac{\sqrt{36}}{2} \\ &= \pm\frac{6}{2} \\ x &= -3 \qquad \text{and} \qquad x = 3 \end{aligned}$$

Therefore, the solutions of the equation are $x = -3$ and $x = 3$. The same results are obtained by factoring.

Now lets try a more complicated quadratic equation.

EXAMPLE 3

Solve $0 = x^2 + x - 3$.

SOLUTION

First observe that $a = 1$, $b = 1$, and $c = -3$. The solutions are:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1 + 12}}{2} \\ &= \frac{-1 \pm \sqrt{13}}{2} \end{aligned}$$

Therefore the solutions are $x = -\frac{1}{2} - \frac{\sqrt{13}}{2}$ and $x = -\frac{1}{2} + \frac{\sqrt{13}}{2}$.

EXAMPLE 4

Solve the equation $0 = 2x^2 - x + 4$.

SOLUTION

Observe that $a = 2$, $b = -1$, and $c = 4$. The solutions to the equation are:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{(-1)^2 - 4(2)(4)}}{2(2)} \\ &= \frac{-1 \pm \sqrt{-31}}{4} \end{aligned}$$

There are no real solutions, because the square root of a number does not make sense as a real number. However, it is possible to make sense of this situation if we allow the use of complex numbers, which are discussed in other modules. If we allow complex numbers, then there are two complex solutions to the quadratic equation.

Graphically, the lack of solutions means that the parabola that is the graph of the function $y = 2x^2 - x + 4$ does not intersect the x-axis.

EXAMPLE 5

Solve the equation $0 = x^2 - 6x + 9$.

SOLUTION

Observe that $a = 1$, $b = -6$, and $c = 9$. The solutions to the equation are:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)} \\ &= \frac{6 \pm \sqrt{0}}{2} \\ &= 3 \pm 0 \\ &= 3 \end{aligned}$$

Therefore there is only one solution, $x = 3$.

KEY IDEA

From the examples above, you can see that the quadratic formula works for all types of quadratic equations. Some quadratic equations have two solutions, some have one solution, and some have no solutions.

INVESTIGATION

Explore this on your own!

Recall from your previous studies that the vertical line through the vertex is the axis of symmetry of the parabola. One consequence of this is that the x-coördinates of the zeros, if there are any, are the same distance from the x-coördinate of the vertex.

You might like to explore how to prove that these statements are correct using the quadratic formula.

PRACTICE

(Answers below.)

1. Determine the zeros, if there are any, for each quadratic function.

   (a) $y = 3x^2 - 12$ (b) $x^2 = 2y + 5x$ (c) $y + 4 = x^2 + 2$ (d) $y + 2 = x^2 + 4$

Answers: 1. (a) $x = 2$ and $x = -2$ (b) $x = 0$ and $x = 5$; (c) $x = \sqrt{2}$ and $x = -\sqrt{2}$ (d) No real solution.

RECAP OF FOCUS QUESTION

Recall the focus question, which was asked earlier in the lesson.

Robin Hood is in a "longest arrow in the air" competition. If Robin's arrow has an initial upward velocity[a] of 50 m/s and reaches a maximum height of 129 m, will he beat Will Scarlet, whose arrow was in the air for 5 seconds, assuming no air resistance? The acceleration due to gravity is 9.8 m/s^2 downward.

SOLUTION

The height y of Robin's arrow above the ground can be modelled by the quadratic function

$$y = -4.9t^2 + 50t + c$$

where the value of c is not very clear at the moment. In fact, how did we get the other coefficients of the quadratic function?

Well, you might recall from a physics class (and don't worry if you have never studied this) that the coefficient of t^2 is *half* of the acceleration due to gravity, with the negative sign used because the acceleration is downward, and we measure the value of y upward. Also, the coefficient of t is the initial upward velocity. The coefficient c represents the initial height of the arrow, but unfortunately we were not given this information. How can we proceed?

If we can determine the value of c, then we'll be able to solve the problem. Is there any other information that was given that can help us determine the value of c. Re-reading the question, note that the maximum height of the arrow is 129 m. Recall from Module 11A2.4 (Maximum or Minimum Values for a Quadratic Function) that the y-coordinate of the vertex corresponds

[a] Robin has clearly ingested his vitamins today (we can be certain of this because performance-enhancing drugs were not available in Robin Hood's era).

to the maximum or minimum value of a quadratic function. We know our quadratic function has a maximum because the coefficient of t^2 is negative, which means the graph is a parabola opening down. So if we can determine an expression for the y-coördinate of the vertex, and then equate the expression to 129, we hope to be able to solve for the value of c.

Let's carry out this strategy. First we need to complete the square:

$$\begin{aligned}
-4.9t^2 + 50t + c &= -4.9\left[t^2 - \frac{50}{4.9}t - \frac{c}{4.9}\right] \\
&= -4.9\left[t^2 - \frac{50}{4.9}t - \frac{c}{4.9} + \left(\frac{25}{4.9}\right)^2 - \left(\frac{25}{4.9}\right)^2\right] \\
&= -4.9\left[t^2 - \frac{50}{4.9}t + \left(\frac{25}{4.9}\right)^2 - \frac{c}{4.9} - \left(\frac{25}{4.9}\right)^2\right] \\
&= -4.9\left[t^2 - \frac{50}{4.9}t + \left(\frac{25}{4.9}\right)^2\right] - 4.9\left[-\frac{c}{4.9} - \left(\frac{25}{4.9}\right)^2\right] \\
&= -4.9\left[t - \frac{25}{4.9}\right]^2 + c + \frac{25^2}{4.9}
\end{aligned}$$

From the previous equation, we can read off that the coördinates of the vertex are $t = 25/4.9$ and $y = c + 25^2/4.9$. Equating the y-coördinate of the vertex to 129, we obtain the value of c:

$$\begin{aligned}
c + \frac{25^2}{4.9} &= 129 \\
c &= 129 - \frac{25^2}{4.9} \\
&= 1.449
\end{aligned}$$

Therefore, the initial height of the arrow is about 1.45 m. This means that the function that models the height of the arrow at time t is

$$y = -4.9t^2 + 50t + 1.449$$

To determine the time at which the arrow hits the ground, set $y = 0$ and solve for t. One way to do this is to use the quadratic formula, noting that $a = -4.9$, $b = 50$, and $c = 1.449$:

$$\begin{aligned}
t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-50 \pm \sqrt{50^2 - 4(-4.9)(1.449)}}{2(-4.9)} \\
&= \frac{-50 \pm \sqrt{2528.4004}}{-9.8} \\
&= \frac{-50 \pm 50.2832}{-9.8} \\
&= \frac{0.2832}{-9.8} \quad \text{and also} \quad \frac{-100.2832}{-9.8} \\
&\approx -0.03 \quad \text{and also} \quad 10.2
\end{aligned}$$

The negative time value can be rejected as not being relevant for our problem. Thus, the arrow stays in the air for a little more than 10 s, which easily beats Will Scarlet's time.

DISCUSSION PROBLEM

The following problem is open in the sense that there may be no definitive solution. Unlike typical textbook exercises, real-life problems rarely have cut-and-dried solutions. Discuss this problem with classmates or friends, then do your best to come up with a reasonable solution, and be prepared to identify and defend the assumptions you make.

Meaning of the Zeros

In the solution of the Focus Problem, we rejected the negative solution as not being relevant for the problem. But nevertheless, the negative solution has a meaning, and can be interpreted even in the context of this problem.

What does the negative solution mean?

And another thing: Some sources claim that "time cannot be negative," and use this as a bogus reason for rejecting all negative times without further thought. This really is thoughtless. Explain how to interpret negative times; this may help you to answer the question in the previous paragraph.

WWW

- What we did: We learned the quadratic formula, which allows us to solve any quadratic equation.
- Why we did it: Being able to solve various types of equations is a key problem-solving tool, so any types of equations that we learn how to solve expands our ability to solve problems.
- What's next: You can explore how to solve other types of equations, and also learn how to use computer software (and the relevant algorithms) to approximate the solutions of equations that cannot be solved exactly by formula.

EXERCISES

1. Determine the zeros of each quadratic function.

(a) $y = 3x^2 - 1 + 4x$ (b) $y = 2x^2 + 6x - 5$ (c) $y = x^2 + 9$ (d) $y = 4x^2 + 16x - 36$

2. A ball is thrown straight up with an initial velocity of 25 m/s, and reaches a maximum height of 62 m. How long will the ball remain in the air if there is no air resistance?

CHALLENGE PROBLEM

A photograph has a length that is $\frac{4}{3}$ its width. It is to be enlarged to have an area of 192 cm^2, with the proportions remaining the same. What will be the dimensions of the enlargement?