BROCK UNIVERSITY MATHEMATICS MODULES

12B1.1: Introduction to Radian Measure

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WWW

- What it is: A unitless angle measure that is an alternative to degree measure.
- Why you need it: Using radians instead of degrees simplifies certain formulas in mathematics and in applications of mathematics to other branches of science. It also gives us further perspective on the formulas for the circumference and area of a circle.
- When to use it: Radian measure simplifies derivative formulas in calculus, and also simplifies formulas in physics that connect linear measures with angular measures.

PREREQUISITES

Before you tackle this module, make sure you have completed these modules:

Circumference of a Circle, Degree Measurement, Unit Conversions

WARMUP

Before you tackle this module, make sure you can solve the following exercises. If you have difficulties, please review the appropriate prerequisite modules.

(Answers below.)

- 1. Determine the third angle of a triangle, given the first two.

 - (a) $A = 60^{\circ}$, $B = 30^{\circ}$ (b) $A = 50^{\circ}$, $B = 50^{\circ}$

 - (c) $A = 90^{\circ}$, $B = 50^{\circ}$ (d) $A = 35^{\circ}$, $B = 42^{\circ}$
- 2. Determine the fourth angle of a quadrilateral, given the first three.

 - (a) $A = 60^{\circ}$, $B = 30^{\circ}$, $C = 110^{\circ}$ (b) $A = 167^{\circ}$, $B = 32^{\circ}$, $C = 83^{\circ}$
- 3. Given the number of sides of each closed shape, determine the sum of the interior angles.

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- (a) 3 (b) 5 (c) 6
- 4. For each radius of a circle, calculate the circumference.
- (a) r = 1 (b) r = 3 (c) r = 2.41

- 5. For each circumference of a circle, calculate the radius.
 - (a) C = 6.28 (b) C = 9.58 (c) C = 2
- 6. Convert each length into the given unit.
 - (a) 6 cm to mm
- (b) 15 mm to m (c) 2.5 km to cm

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1.(a) 90^{\circ} (b) 80^{\circ} (c) 40^{\circ} (d) 103^{\circ} 2.(a) 160^{\circ} (b) 78^{\circ} 3.(a) 180^{\circ} (b) 540^{\circ} (c) 720^{\circ} 4.(a)
2\pi (b) 6\pi (c) \approx 15.14 5.(a) \approx 1 (b) \approx 1.52 (c) \approx 0.32 6.(a) 60 mm (b) 0.015 m (c) 250 000 cm
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Introduction

In this module, you'll learn what a radian is, and how to use radians to measure angles. Radian measure is an alternative measure for angles and can helpfully simplify some formulas in calculus and physics. All of you have used radian measurements already in previous math courses, perhaps without even knowing it! We'll look at this a little later in the lesson.

One of the great advantages of radian measure is that it is a unitless measure.

In previous modules, we used degrees as a measurement for all angles. Now, we'll learn how to use radian measure to answer questions such as this:

FOCUS QUESTION

To help you understand an important aspect of this lesson, focus your attention on this question, which will be answered towards the end of the lesson.

Alice created a bat-swinging robot to win a home-run challenge at a local baseball tournament. This robot pivots a 0.85 m long bat about the end of its handle. The tip of the bat travels at exactly 12 m/s for 0.2 seconds. Through what angle does the bat turn?

Measuring an Angle in Degrees

Let's begin by discussing how angles are measured. Suppose we wish to measure the following angle:

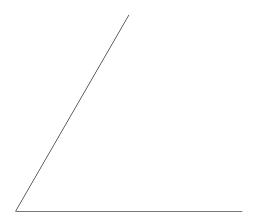


Figure 1: How can we measure this angle?

What we normally do is cut the angle up into equally small portions, each called a *degree*, as seen in the following picture:

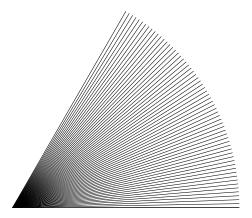


Figure 2: We can see that this is a 60° angle, because it contains 60 small 1° angles.

We could then count up how many small 1° angles are contained in the large angle; in this case, there are 60 small angles in the large angle, so we know the measure of the large angle is 60°. In this way, we can measure angles to the nearest degree. If we need to measure an angle more accurately, we could cut each of the small 1° angles into even smaller parts. The same idea is used for linear measures, where the centimetres on a ruler are divided into the smaller millimetres.

There are many other units that are sometimes used for measuring angles. However, in mathematics and the sciences, the most commonly used measures are degrees and a unitless measure called a radian, which we'll now discuss.

Measuring an Angle in Radians

One way of understanding radian measure is to imagine yourself starting at the vertex of the previous angle and walking along one edge. You walk a certain distance, which we'll denote by r. Then, you change direction and walk a distance s along a circular arc across to the other edge, as shown below:

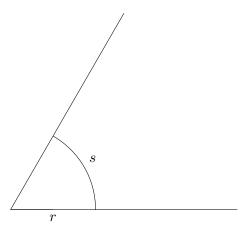


Figure 3: Measuring an angle based on the arc length s and the radius r.

Notice that the bigger the angle gets, the bigger the distance s will become. Thus, we can use the distance s along the arc as a measure of the angle. This will be the basis for our explanation of radian measure.

But how? you may ask. You can pick any random length of r and get a different length of s, so how can this be a constant measurement for an angle?

Well, the answer is simple, as you can see for yourself by investigation: Time to grab some paper, pencils, ruler, string and a protractor (or a compass for you fun-loving circle drawers).

Draw any angle that you know the measure of in degrees; I like 60° angles, so that's what I've drawn here. Make the lines, say, 10 cm long, as shown below:

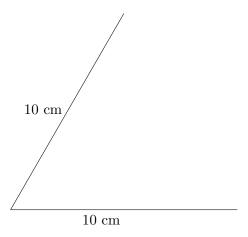


Figure 4: A 60° angle with side lengths of 10 cm.

From there, mark a few different lengths; lets say at 1 cm, 4 cm, and 9 cm. Then use your protractor and draw a circular arc connecting one side to the other, making sure that the distances are the same on both lines, as shown below:

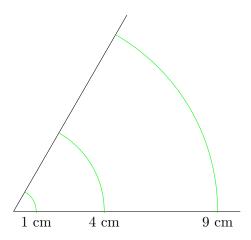


Figure 5: The same 60° angle as in the previous figure, with arcs at 1 cm, 4 cm, and 9 cm.

This is where your string comes into play. Take your string, and to the best of your ability, place it over the arc and mark on it how far the curve goes. Do this for each arc and then measure the length of each portion of string. Do you notice anything special about these numbers?

Length of Radius	Length of Arc
1 cm	$1.05~\mathrm{cm}$
4 cm	4.19 cm
9 cm	9.42 cm

If you divide the length of the arc by the matching radius, you obtain (approximately) the same number each time! This value is the measure of the angle in radians. For the 60° angle, your measured value should be close to the true value, which is approximately 1.05 radians.

If we could get *exact* measurements, we would notice that the ratio of arc length to the radius is constant for a given angle, and thus can be used as a measure for the angle. This leads us to the following definition:

DEFINITION

The measure of an angle in radians is obtained by dividing the length of an arc constructed on the angle by the radius of the arc. That is,

$$\theta = \frac{s}{r}$$

where θ is the angle in radians, s is the arc length, and r is the length of the radius. Note that the radius and arc length must be measured in the same units.

Let's use the definition of radian measure in a few simple examples:

EXAMPLE 1

Given s and r, calculate each angle in radians:

(a)
$$r = 1 \text{ cm}, s = 1 \text{ cm}$$

(b)
$$r = 25$$
 cm, $s = 0.5$ m

SOLUTION

For (a), we want to make sure that the units of the radius and arc length are the same. Notice both are in cm, meaning we don't need to convert any of the units. So all you do now is substitute the given values into the formula to get:

$$\theta = \frac{s}{r}$$

$$= \frac{1 \text{ cm}}{1 \text{ cm}}$$

$$= 1 \text{ radian}$$

Therefore, with a radius and arc length of 1 cm, the measure of the angle is 1 radian.

For (b), we can see that the units of the radius and the arc length are different. So our first step is to convert the lengths so they are measured in the same unit. In centimetres, the lengths are: r = 25 cm and s = 50 cm.^a Now that the lengths are in the same unit, just substitute them into the formula to get:

$$\theta = \frac{s}{r}$$

$$= \frac{50 \text{ cm}}{25 \text{ cm}}$$

$$= 2 \text{ radians}$$

Thus, the angle has a measure of 2 radians.

EXAMPLE 2

Given θ and r, determine the arc length:

$$\theta = 3$$
 , $r = 1$ m

SOLUTION

Notice that θ has no units (that is, it is in radian measure), which makes this calculation easy. If θ were measured in degrees, an additional step would be needed to convert the unit to radians.^a Because no unit conversion is needed here, we can substitute the given values into our equation and obtain:

$$\theta = \frac{s}{r}$$

$$s = r\theta$$

$$= 3 \cdot 1 \,\mathrm{m}$$

$$s = 3 \,\mathrm{m}$$

Remember that the angle is unitless, and the radius has a unit of metres, so the arc length s also has metres as its unit.

The previous two examples remind us of the following key idea:

KEY IDEA

A radian, although named, has no units. That is, 1 cm multiplied by 1 radian is just 1 cm. Looking at the previous example, we can see that we multiplied the angle by the radius and obtained the arc length. And for our arc length, we obtained the same unit of length as the radius, confirming that the radian is a unitless angle measure.

Now it's time to try a few questions out on your own.

^aFor unit conversions, see Module ***

^aYou can see an example of this in Module ***.

PRACTICE

(Answers below.)

1. Given the radius r and the arc length s, calculate the angle θ in radians.

(a)
$$r = 1$$
 cm, $s = 2$ cm (b) $r = 3$ m, $s = 2$ cm (c) $r = 7$ km, $s = 60$ m

2. Given the angle θ in radians and the radius r, calculate the arc length s.

(a)
$$\theta = 1$$
, $r = 2$ cm (b) $\theta = 3.2$, $r = 10$ km (c) $\theta = 10$, $r = 6.7$ m

3. Given the angle θ in radians and the arc length s, calculate the length of each radius r.

(a)
$$\theta = 4.3$$
, $s = 2$ cm (b) $\theta = 9.2$, $s = 10$ km (c) $\theta = 0.35$, $s = 6.7$ m

Answers: 1. (a) 2 radians (b) $1/150 \approx 0.007$ radians (c) $6/700 \approx 0.00857$ radians 2. (a) 2 cm (b) 32 km (c) 67 m 3. (a) 0.465 cm (b) 1.087 km (c) 19.143 m

MAKING CONNECTIONS

Remember when I told you that you've used radians before and possibly didn't realize it? Let's now look at the equation you used for circumference of a circle in a different light.

$$C = 2\pi r$$

Let's re-arrange a bit and see what we can get. If we divide both sides by r, we obtain:

$$\frac{C}{r} = 2\pi$$

Does this not look extremely familiar? Let's compare it with the formula

$$\frac{s}{r} = \theta$$

How about now? Notice anything ... radical about it? Isn't it amazing how something that we've used for so long is also related to something brand new? Mathematics is fantastic that way ... *drifts off into a memory of drawing circles and calculating fractions*...oh ... sorry about that. Let's get back on track, now.

Comparing the two formulas suggests that when the arc length is the entire circumference of the circle, the angle is 2π . We can see this similarly by letting $s = 2\pi r$ in the previous formula:

$$\theta = \frac{s}{r}$$

$$= \frac{2\pi r}{r}$$

$$= 2\pi$$

Seemed quick, did it not? From this we can conclude that any circle, no matter the radius, has a radian measure of 2π . Note that we usually leave the π term in with radian measure, and this will become clear in later modules.^a

^aThis will be discussed in Module ***.

KEY IDEA

We just calculated that the angle in a circle is 2π radians. If we express this as a decimal number, we get:

$$2\pi = 6.283185... \approx 6$$

This means that the angle of a circle is a little more than 6 radians. We also know that the angle in a circle is equal to 360° . This means that one radian is approximately equal to $\frac{360^{\circ}}{6} = 60^{\circ}$.

Let's try a few examples of some word problems that make use of the definition of radian measure.

EXAMPLE 3

A race car drives at a constant speed around a semi-circular bend that has a radius of 150 m. Calculate the car's speed if the time needed to drive around the bend is (a) 5π s, and (b) 30 s.

SOLUTION

Because the car drives at a constant speed, we can calculate the speed by dividing the distance travelled by the time. Since the car's path is a circular arc, we can use the formula $s=r\theta$ to calculate the distance. We know the radius is r=150 m, and the angle is $\theta=\pi$, because a semi-circle has half of the angle of a full circle. Substituting these values into the formula $s=r\theta$, we obtain:

$$s = r\theta$$
$$s = 150\pi \text{ m}$$

This shows that the distance that the driver travels is 150π m. Now that we know the distance the driver traveled, we can compute the speeds as follows:

(a) speed =
$$\frac{\text{distance}}{\text{time}}$$

= $\frac{150\pi \text{ m}}{5\pi \text{ s}}$
= 30 m/s

(b) speed =
$$\frac{\text{distance}}{\text{time}}$$

= $\frac{150\pi \text{ m}}{30 \text{ s}}$
= $5\pi \text{ m/s}$
 $\approx 15.71 \text{ m/s}$

If desired, you can convert the units of the speeds to km/h (see Module *** for a reminder about how to do this). The results are:

(a)
$$30 \text{ m/s} = 30 \frac{\text{m}}{\text{s}} \cdot \frac{1 \text{ km}}{1000 \text{ m}} \cdot \frac{3600 \text{ s}}{1 \text{ h}}$$

= 108 km/h

(b)
$$5\pi \text{ m/s} = 5\pi \frac{\text{m}}{\text{s}} \cdot \frac{1 \text{ km}}{1000 \text{ m}} \cdot \frac{3600 \text{ s}}{1 \text{ h}}$$

 $\approx 56.5 \text{ km/h}$

PRACTICE

(Answers below.)

- 4. A rubber duckie is swirling down a bathtub drain before Jenny plugs it. Jenny measures that the duck travels 30 cm overall. If the duck is 15 cm away from the drain at all times, through what angle does the duck travel?
- 5. A few boys like to swing over a river from a rope hanging from a tree. The record swing so far is through an angle of $\pi/3$ radians. If the rope is 6 m long, how far does the bottom of the rope travel?

Answers: 4. 2 radians 5. 2π m ≈ 6.28 m

RECAP OF FOCUS QUESTION

Recall the focus question, which was asked earlier in the lesson.

Alice created a bat-swinging robot to win a home-run challenge at a local baseball tournament. This robot pivots a 0.85 m long bat about the end of its handle. The tip of the bat travels at exactly 12 m/s for 0.2 seconds. Through what angle does the bat turn?

SOLUTION

The given information is:

$$r = 0.85 \text{ m}$$

 $v = 12 \text{ m/s}$
 $t = 0.2 \text{ s}$

Multiplying the speed v by the time t tells us how far s the tip of the bat travels in an arc:

$$s = v \cdot t$$

$$= 12 \text{ m/s} \cdot 0.2 \text{ s}$$

$$= 2.4 \text{ m}$$

So the arc through which the bat travels is 2.4 m long. Now, substituting this into the equation for the angle, we get:

$$\theta = \frac{s}{r}$$

$$= \frac{2.4 \text{ m}}{0.85 \text{ m}}$$

$$\approx 2.82$$

This means that the angle that the bat moves through is approximately 2.82 radians. How does this compare to a typical swing for a human batter?

DISCUSSION PROBLEM

The following problem is open in the sense that there may be no definitive solution. Unlike typical textbook exercises, real-life problems rarely have cut-and-dried solutions. Discuss this problem with classmates or friends, then do your best to come up with a reasonable solution, and be prepared to identify and defend the assumptions you make.

Kicking a soccer ball

You might like to try this out on your own with a camera and attempt to estimate the result in your specific case. Then you might try to estimate the result in general.

Imagine a soccer player getting ready for a free kick. He moves towards the ball, and when close enough, he plants one foot. From this moment on, perhaps the player's hips are almost stationary, but the non-planted leg is pivoting about its hip. Besides rotating about the hip, the lower part of the kicking leg also experiences rotation about the knee.

Make some reasonable assumptions about the motion of the kicking leg after the other leg has planted. For example, you might assume that the foot of the kicking leg starts 0.5 m off the ground, and rotates at twice the rate of the knee. By the time the foot hits the ball (which is next to the planted foot) the knee locks, so that the leg moves rigidly after impact. If, on average, the foot is moving at 23 m/s, and the foot finally stops 1.2 m above the ground, how far did the foot move? Is there enough information given to solve the problem, or is further information needed? If there is missing information, then make reasonable assumptions in approximating it.

WWW

- What we did: We learned about radian measure and how to use it.
- Why we did it: Since it is a unitless measure, we can use radian measure for calculating the lengths of circular arcs, speeds, and other quantities in natural units.
- What's next: Converting from degrees to radians and back.

EXERCISES

6. Given the radius r and the arc length s, calculate the angle θ .

(a)
$$r = 2.3 \text{ m}$$
, $s = 7 \text{ m}$ (b) $r = 8 \text{ cm}$, $s = 6 \text{ mm}$ (c) $r = 160 \text{ m}$, $s = 1.2 \text{ km}$

7. Given the arc length s and the angle θ , calculate the radius r.

(a)
$$s=4$$
 m, $\theta=\pi$ radians (b) $s=12$ km, $\theta=6$ radians (c) $s=1248$ m, $\theta=\frac{3\pi}{2}$ radians

- 8. A golfer likes to take huge power swings at his ball to try to drive it as far as possible. Knowing that his club is 1.2 m long and that he swings 7/8 of a rotation, how far does the head of his club travel? Assuming the average speed of the head of the club is 90 km/h, how long does it take to complete the rotation in seconds?
- 9. A merry-go-round makes 4 rotations per minute. Assuming you have rotated exactly 17 times, and your average speed is 2π m/min, how far from the centre of the merry-go-round are you?
- 10. Every day after school, Sally goes to the park to play on the swings. She tries to get higher and higher each day. To help her out, Sally's mother placed a device that displays how large an angle she swings through. Yesterday, Sally swung through an angle of $5\pi/8$ radians. If the length of the swing is 2 m, how much farther did she swing today if she swung through an angle of $3\pi/4$ radians?

CHALLENGE PROBLEM

Patti and Bill are playing tether-ball. The tether-ball hangs from the top of a pole by a rope. The two players, who are directly opposite each other, hit the ball back and forth until the ball wraps all the way around the pole.

The height of the pole is 3 m and the length of the rope connecting the ball to the top of the pole is 2.2 m. Bill hits a weak shot towards Patti. Assuming that the ball remains at a constant height of 1 m above the ground, and at a constant distance from the pole, how far does the ball move (in metres) before Patti hits it back?



http://en.wikipedia.org/wiki/Tetherball (downloaded 04/06/10)