

BROCK UNIVERSITY MATHEMATICS MODULES

11A2.1: Determining Zeros for a Quadratic Function

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WWW

- What it is: A zero is a value for which the graph of a function intersects the x -axis.
- Why you need it: Determining the zeros of a function is often meaningful, and quadratic functions appear frequently in applications.
- When to use it: When you need to determine the x -intercepts of a quadratic function.

PREREQUISITES

Before you tackle this module, make sure you have completed these modules:

Functions, Function notation, Solving linear equations, Square roots, Solving equations of the form $x^2 = 4$

WARMUP

Before you tackle this module, make sure you can solve the following exercises. If you have difficulties, please review the appropriate prerequisite modules.

(Answers below.)

1. Identify the type of function listed (linear, quadratic, cubic, quartic) :

(a) $f(x) = 4x^3 + 2x^2 - x + 9$ (b) $y = -\frac{1}{2}x + 25$ (c) $y = x^2 + 5x - 4$ (d) $y = x^4 - 6x^3 + 2x^2 - 4x - 9$

2. Determine each value for the functions $f(x) = 2x - 7$, $g(x) = x^2 - 5$, and $h(x) = 3x^2 - 4x - 1$.

(a) $f(2)$ (b) $g(3)$ (c) $h(-2)$

3. Solve each linear equation. (a) $x + 2 = -5$ (b) $3x - 4 = 0$ (c) $2x + 5 = -8$

4. Use your calculator (or otherwise) to calculate each value. (a) $\sqrt{4}$ (b) $\sqrt{27.04}$ (c) $\sqrt{17}$

5. Determine the value of x in each case. (a) $x^2 = 4$ (b) $x^2 = 27.04$ (c) $x^2 = 17$

Answers: 1.(a) cubic (b) linear (c) quadratic (d) quartic 2.(a) -3 (b) 4 (c) 19 3.(a) $x = -7$ (b) $x = 4/3$ (c) $x = -13/2$ 4.(a) 2 (b) 5.2 (c) ≈ 4.12 5.(a) $x = \pm 2$ (b) $x = \pm 5.2$ (c) $x = \pm\sqrt{17} \approx \pm 4.12$

Introduction

In this module, you'll learn what the zeros of a quadratic function are, and how to find the zeros of simple quadratic functions. The zeros, also known as x -intercepts, of a quadratic function are the points where the graph of the function crosses the x -axis. Another way to think of this is that the zeros of a quadratic function are the x -values for which the matching y -value of a quadratic function is equal to zero.

Zeros of quadratic function often have meaning in the contexts of various problems. In general, many applications of mathematics involves solving equations, and quadratic equations are fundamental; determining the zeros of a quadratic function amounts to solving a quadratic equation.

FOCUS QUESTION

To help you understand an important aspect of this lesson, focus your attention on this question, which will be answered towards the end of the lesson.

A sailor stranded on a desert island has one flare to shoot into the sky to get the attention of incoming boats or planes. The sailor wants to know how long the flare will stay in the air when he shoots it. He knows that gravity accelerates objects down at about 10 m/s^2 and that the flare gun shoots the flares up with an initial speed of 100 m/s . This means that the height y (in metres) of the flare t seconds after it is launched can be approximated by the formula $y = 100t - 10t^2$.

- (a) How long does the flare stay in the air?
- (b) How long does the flare stay above a height of 50 m ?

Because the zeros of a quadratic function are x -values for which the function values are zero (i.e., $f(x) = 0$), one can read the zeros of a quadratic function from a graph. At least one can do so approximately, to a degree of approximation depending on how good our eyes are and how accurately the graph is plotted. For example, consider the graph of the quadratic function in Figure 1. It seems clear from the graph that the zeros of the plotted function are approximately $x = -1$ and $x = 1$.

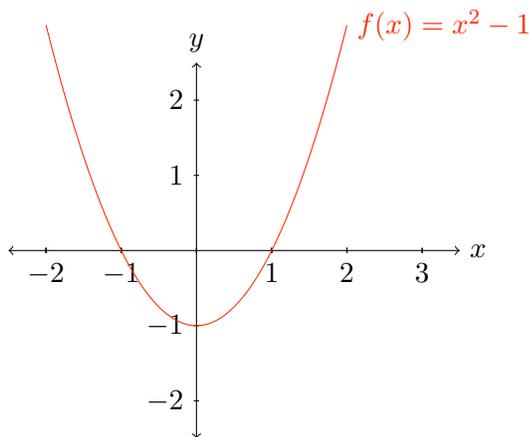


Figure 1: The zeros of this quadratic function can be read from the graph; they seem to be (as far as our eyes can tell) $x = -1$ and $x = 1$.

However, what if the graph has zeros at values of x that are not integers? Then it would be harder to approximate the zeros by eye. And sketching graphs is a pain that one would rather

avoid unless it's necessary. For both of these reasons it would be nice if we had a way of calculating the zeros algebraically.

The key is to solve the equation $f(x) = 0$ for x , if it's practical (and possible) to do so. In this module we'll tackle situations where it is relatively easy to solve such equations; in subsequent modules, we'll tackle more complicated cases, where factoring is requiring. Sometimes factoring is not practical either, in which case we need to use completing the square, or the quadratic formula, topics that are studied in subsequent modules.

Let's now return to the function in Figure 1. To determine the zeros by formula, solve the equation $f(x) = 0$:

$$\begin{aligned}f(x) &= 0 \\x^2 - 1 &= 0 \\x^2 &= 1 \quad (\text{adding 1 to each side of the equation in the previous line}) \\x &= \pm 1 \quad (\text{when you take the square root, the result could be positive or negative})\end{aligned}$$

This algebraic procedure leaves us with no doubt: Our eyes did not deceive us, and the zeros we read off the graph in Figure 1 are indeed correct.

KEY IDEA

Square roots

A convention about square roots sometimes causes confusion, so let's make sure we've got this one straight. When you see a radical expression that is written down, such as $\sqrt{4}$, the convention is that this means the *positive* square root of 4. That is, $\sqrt{4} = 2$, where the negative value is *not included*. Similarly, the graph of the function $y = \sqrt{x}$ is just the positive half of the parabola, not the negative half, whereas the graph of the relation $y^2 = x$ is the full parabola opening to the right. This justifies our referring to $y = \sqrt{x}$ as a function, because its graph passes the vertical line test.

However, if you are solving an equation, and *you* take the square root of both sides of the equation, then you must remember to consider the possibility that the result might be positive or negative. For example, to solve the equation $x^2 = 4$, take the square root of each side to obtain:

$$\begin{aligned}x &= \pm\sqrt{4} \\&= \pm 2\end{aligned}$$

This correctly expresses the fact that both $2^2 = 4$ and $(-2)^2 = 4$.

Of course, it's always wise at the very end to check your solutions back in the original equation, as we just did, because solving equations involving square roots sometimes produces what are called *extraneous* solutions. An extraneous solution is not a solution to the original equation at all, even though no mistakes were made in the solution process; rather, it is a solution to an equation that is related to the original equation, but not exactly the same. (The mathematical usage of the term extraneous relies on its dictionary definition "irrelevant" or "external to the problem.")

In conclusion, even though you consider the possibility of positive and negative solutions when *you* take the square root, you must always check in the end to see if either, both, or none of the results really is a solution to the original equation.

EXAMPLE 1

Determine the zeros of each function.

(a) $f(x) = x^2 - 4$ (b) $g(x) = x^2$ (c) $h(x) = x^2 + 1$

SOLUTION

(a) Following the procedure outlined earlier in this module, set $f(x) = 0$ and then solve for x :

$$\begin{aligned}x^2 - 4 &= 0 \\x^2 &= 4 \quad (\text{add 4 to each side of the previous equation}) \\x &= \pm 2 \quad (\text{take the square root of each side of the previous equation})\end{aligned}$$

Finally, check both of the values to make sure that they really are solutions to the equation:^a

$$\begin{aligned}(-2)^2 &= 4 \quad \text{YES!} \\2^2 &= 4 \quad \text{“YES! And the foul!”}\end{aligned}$$

Thus, the function $f(x) = x^2 - 4$ has two zeros, at $x = -2$ and $x = 2$.

(b) Using the same procedure as in Part (a), we get:

$$\begin{aligned}x^2 &= 0 \\x &= 0 \quad (\text{take the square root of each side of the previous equation})\end{aligned}$$

Thus, the function $g(x) = x^2$ has one zero, at $x = 0$.

(Interesting, isn't it? In this case there is only one zero, whereas for the function in Part (a) there were two zeros. I wonder what this looks like on a graph.^b)

(c) Using the same procedure as in Parts (a) and (b), we get:

$$\begin{aligned}x^2 + 1 &= 0 \\x^2 &= -1 \quad (\text{subtract 1 from each side of the previous equation}) \\x &= \pm\sqrt{-1} \quad (\text{take the square root of each side of the previous equation})\end{aligned}$$

Hold on a minute, here! You might recall that $\sqrt{-1}$ is not a real number. That is, there is no real number such that when you square it you get -1 . Does this make sense? Let's think about it: If you square a positive number, the result is positive, because a positive number times a positive number gives a positive number. Similarly, if you square a negative number, the result is also positive, because a negative number times a negative number is also positive. If you wish to consider 0 a special case, then $0^2 = 0$. There is no way to square a real number and get a negative number as a result.^c

The conclusion is that this equation has NO solutions, and so the function $h(x) = x^2 + 1$ has no zeros. What must its graph look like?^d

^aRemember Marv Albert?

^bSketch the graph and see for yourself!

^cHowever, this can happen in the complex number system, and other number systems. Check out the modules on complex numbers to begin exploring this delightful area of mathematics.

^dSketch the graph and see for yourself!

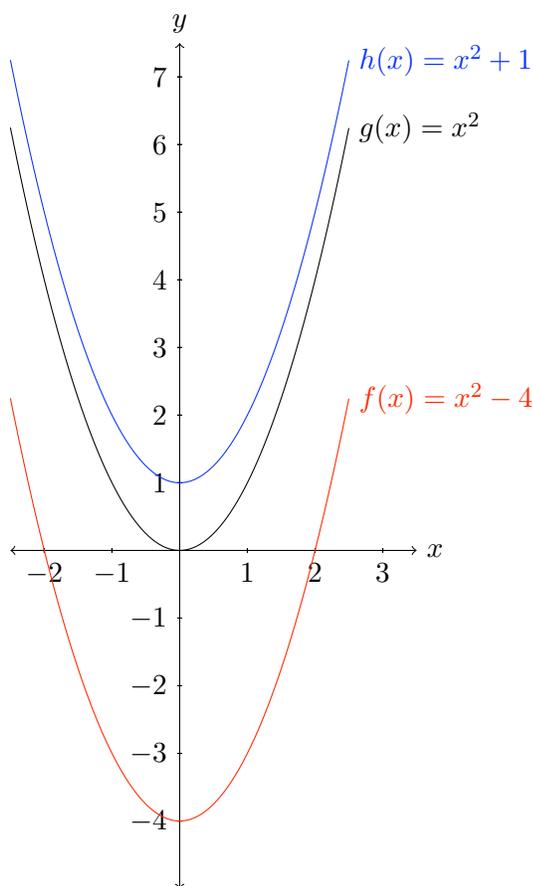


Figure 2: Note that the graph of g has one zero, at $x = 0$. The graph of h can be obtained from the graph of g by translating up by 1 unit. Similarly, the graph of f can be obtained from the graph of g by translating down by 4 units.

One of the things that we have learned from the previous example is that a quadratic function might have no zeros, one zero, or two zeros. (Could a quadratic function have more than two zeros?) You can make sense of this from a graphical perspective by examining the graph in Figure 2, and reading the explanation in the following paragraph.

Thinking in terms of translations helps us to understand why some quadratic functions have two zeros, some have one, and some have none. Start with the graph of f in Figure 2, which has two zeros. Imagine gradually translating the graph of f up; as it moves up, its two zeros will gradually move closer together. Eventually, once the graph of f has moved up far enough, it will reach the position of the graph of g . That is, the zeros of f will have moved far enough that they are actually in the same position. Throughout this process, the y -coordinate of the vertex will have moved gradually up from -4 to its current location of 0 . Continuing to translate the graph up, once the graph passes the position of the graph of g , the y -coordinate of its vertex will be positive. This means that *every* value of y is positive at this point, much as it is with the graph of h . Is it clear that if we continue to translate the graph up there will be no zeros?

In conclusion, for quadratic functions of the form $y = x^2 + c$, the graph will have

- 2 zeros if $c < 0$
- 1 zero if $c = 0$
- NO zeros if $c > 0$

EXAMPLE 2

For quadratic functions of the form $y = -x^2 + c$, for which values of c will the graph have
(a) 2 zeros? (b) 1 zero? (c) NO zeros?

SOLUTION

Let's try to determine the zeros of the function and see what happens. (Maybe it will become clear that it will indeed be possible to determine the zeros, but only under certain circumstances. That's what we're after: the circumstances.)

Following the procedure outlined in the previous example, set $y = 0$ and solve for x :

$$0 = -x^2 + c$$

$$x^2 = c \quad (\text{add } x^2 \text{ to each side of the previous equation})$$

$$x = \pm\sqrt{c} \quad (\text{take the square root of each side of the previous equation})$$

Now notice that:

- If $c = 0$, then the only solution is $x = 0$, because 0 and -0 mean the same thing.
- If $c > 0$, then there are two different solutions, $x = \pm\sqrt{c}$; that is, the two solutions are $x = -\sqrt{c}$ and $x = \sqrt{c}$.
- If $c < 0$, then there are no solutions, because the square root of a negative number makes no sense when we are restricted to using real numbers.

PRACTICE

(Answers below.)

1. Using the method of the previous example, if you wish, verify the conclusion (reached graphically after Example 1) that the graph of a quadratic function of the form $y = x^2 + c$ has 2 zeros if $c < 0$, 1 zero if $c = 0$, and no zeros if $c > 0$.
2. Determine the zeros, if there are any, for each quadratic function.
(a) $y = x^2 - 9$ (b) $y = x^2 + 9$ (c) $y = -x^2 + 4$ (d) $y = -x^2 + 3.72$

Answers: 1. Hint: A possibly confusing part of the argument is that $\sqrt{-c}$ makes sense if and only if $c \leq 0$.
2. (a) $x = -3, x = 3$ (b) No zeros. (c) $x = -2, x = 2$ (d) $x = -\sqrt{3.72}, x = \sqrt{3.72}$

For more complicated quadratic functions, the situation must be similar. We know that the graph of an arbitrary quadratic function is either a parabola opening up or a parabola opening down. Thus, the situation for an arbitrary quadratic function will always be somewhat like the first example, or somewhat like the second example. So you could imagine translating a parabola up or down and seeing the usual things happen: either the two zeros of the parabola get closer together until they merge into one zero, then the parabola ceases to intersect the x -axis so that there are no zeros, or the opposite happens.

The graphical situation, as described in the previous paragraph, is pretty clear, isn't it? However, the algebraic situation is a little more complicated. That is, how do you determine the zeros algebraically in the most general situations? We begin to deal with this question in this module, but the full story involves either factoring the function's formula (if this is practical), or using the quadratic formula if factoring is not practical. Both of these procedures are discussed in the following two modules, Module *** and Module ***.

Let's continue by examining the situation where the formula for the quadratic function is either already factored, or can be easily factored by inspection. Consider the following example.

EXAMPLE 3

Determine the zeros for each quadratic function.

(a) $f(x) = -x(x - 2)$ (b) $f(x) = -x^2 + 2x$

SOLUTION

(a) Follow the procedure of the previous examples by setting $f(x) = 0$ and solving for x :

$$\begin{aligned} -x(x - 2) &= 0 \\ (-1)(x)(x - 2) &= 0 \end{aligned}$$

Now look at the previous equation. We have a product of three factors equalling 0. How can this be? The only way this can occur is if one of the factors in parentheses is equal to 0. However, the first factor, (-1) , is certainly not equal to 0, so one of the other factors must be equal to 0.

There are two possibilities.

CASE 1: Perhaps the (x) factor is equal to 0. That is, $x = 0$. Let's verify that this really is a solution to the original equation:

$$\text{L.S.} = -x(x - 2) = -(0)(0 - 2) = -(0)(-2) = 0 = \text{R.S.}$$

Study the logic of the previous line of equations. We started with the left side of the original equation (abbreviated by L.S.), then substituted what we hoped was a solution, did a bit of algebra, and then saw that the result is really equal to the right side of the original equation. This verifies that $x = 0$ is a solution to the original equation, so 0 is a zero of f .

CASE 2: Perhaps the other factor, $(x - 2)$ is equal to 0. That is, $x - 2 = 0$, from which it follows that $x = 2$. This means that $x = 2$ is also a zero of the quadratic function. To verify this, check by substitution into the original equation:

$$\text{L.S.} = -x(x - 2) = -(2)(2 - 2) = -(2)(0) = 0 = \text{R.S.}$$

Therefore, the zeros of f are $x = 0$ and $x = 2$.

(b) To determine the zeros of f , place it in factored form. The first thing to do, always, when factoring, is to search for common factors. Luckily, this expression has a common factor of x . In fact, let's consider the common factor to be $-x$, and factor the formula for f as follows:

$$f(x) = -x(x - 2)$$

But this is exactly the same expression as in Part (a), and so represents the same function! Therefore, this function has the same zeros as were found in Part (a).

It is interesting to sketch a graph of the function f from the previous example, and to observe on the graph that the zeros really are where we figured they are. See Figure 3.

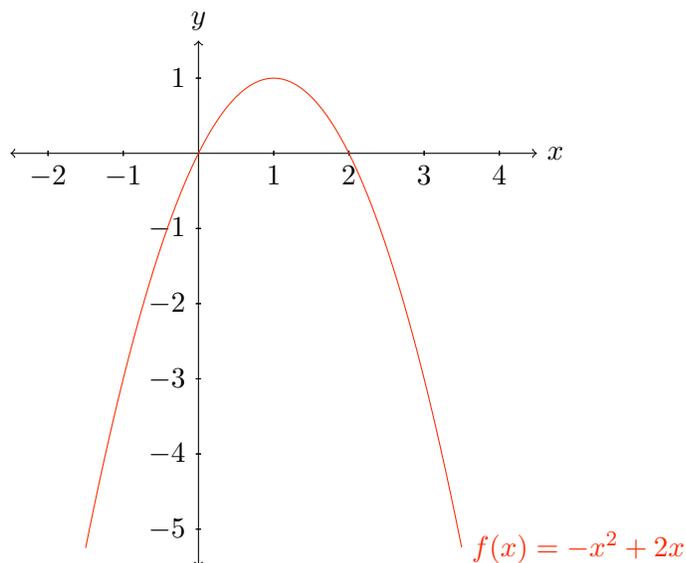


Figure 3: The formula for this function, $f(x) = -x^2 + 2x$, can be placed in factored form as $f(x) = -x(x - 2)$. Note that the zeros on the graph correspond to values of x for which each factor in the factored expression is equal to 0.

KEY IDEA

Notice from the previous example that once the formula for a quadratic function is placed in factored form, there is one zero for each factor. The actual value of the zero is determined by setting each factor equal to 0 and then solving for x .

In the previous example, the factors were (x) and $(x - 2)$, and the matching zeros were $x = 0$ and $x = 2$. In general, for a factor of the form $(x - p)$, there is a matching zero at $x = p$.

Note that if the quadratic function cannot be factored into a product of two linear factors, then it has no real zeros.

PRACTICE

(Answers below.)

3. Determine the zeros, if there are any, for each quadratic function.

(a) $y = x(x - 4)$ (b) $y = -x(x - 3)$ (c) $y = 4x(x + 2)$

4. Determine the zeros, if there are any, for each quadratic function.

(a) $y = x^2 + 3x$ (b) $y = -x^2 - x$ (c) $y = 2x^2 - 3x$

Answers: 3. (a) $x = 0, x = 4$ (b) $x = 0, x = 3$ (c) $x = 0, x = -2$ 4. (a) $x = 0, x = -3$ (b) $x = 0, x = -1$
(c) $x = 0, x = 3/2$

After having completed the previous questions, you might like to sketch the graphs and verify that each zero is matched with a linear factor in the formula for the quadratic function.

EXAMPLE 4

Determine the zeros, if any, for the function $4y = 9x^2 - 6x$.

SOLUTION

Following the usual procedure, we set $y = 0$ and solve for x :

$$\begin{aligned}9x^2 - 6x &= 0 \\x(9x - 6) &= 0\end{aligned}$$

Therefore, either $x = 0$ or $9x - 6 = 0$. In the second case,

$$\begin{aligned}9x - 6 &= 0 \\9x &= 6 \\x &= \frac{6}{9} \\x &= \frac{2}{3}\end{aligned}$$

Therefore the zeros of the function are $x = 0$ and $x = \frac{2}{3}$.

INVESTIGATION

Explore this on your own!

It might be extremely challenging to do so, but I'm wondering whether it's possible to state conditions for which a cubic function (i.e., of the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$) has no zeros, one zero, two zeros, three zeros, and so on. The first step might be to collect some data, using a graphing calculator, or some of the graphing software that is freely available via internet. The next step would be to try to do some sort of calculations that would help you determine a definitive result by reasoning.

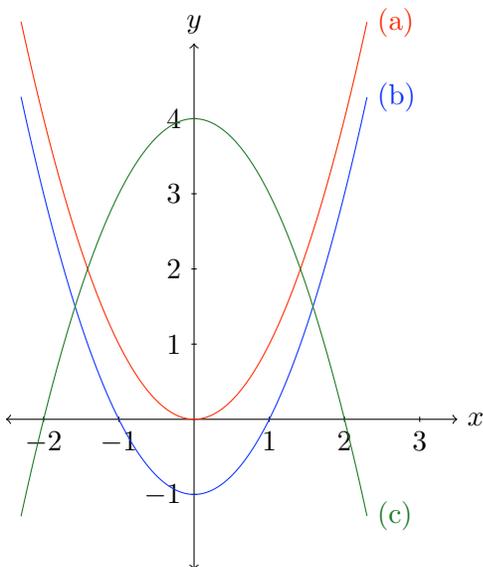
Have fun exploring this! If you obtain satisfying results, you might wish to turn your attention to higher-degree polynomials as a next step.

And once you learn calculus, then the fun will only increase: Then you will have additional tools that will help you explore situations such as this one.

PRACTICE

(Answers below.)

5. Use the graph to state the zeros of each quadratic function.



6. Determine the zeros, if any, for each quadratic function.

(a) $y = 3x^2 - 12$ (b) $x^2 = 2y + 5x$ (c) $y + 4 = x^2 + 2$ (d) $y + 2 = x^2 + 4$

Answers: 5. (a) $x = 0$ (b) $x = -1, x = 1$ (c) $x = -2, x = 2$ 6. (a) $x = -2, x = 2$ (b) $x = 0, x = 5$ (c) $x = \pm\sqrt{2} \approx \pm 1.414$ (d) No solutions.

Now, let's answer the focus question, if you haven't already.

RECAP OF FOCUS QUESTION

Recall the focus question, which was asked earlier in the lesson.

A sailor stranded on a desert island has one flare to shoot into the sky to get the attention of incoming boats or planes. The sailor wants to know how long the flare will stay in the air when he shoots it. He knows that gravity accelerates objects down at about 10 m/s^2 and that the flare gun shoots the flares up with an initial speed of 100 m/s . This means that the height y (in metres) of the flare t seconds after it is launched can be approximated by the formula $y = 100t - 10t^2$.

- (a) How long does the flare stay in the air?
(b) How long does the flare stay above a height of 50 m?

SOLUTION

(a) When the flare is shot into the air, it is at ground level, which we might as well label with a height of $y = 0$. When solving motion problems, we typically represent the time at which

a motion begins as $t = 0$. Both of these conventions are already built into the formula for the height function; you can check this for yourself by verifying (using the formula) that when $t = 0$, the height is $y = 0$.

When the flare returns to the ground, once again $y = 0$. Thus, if we set $y = 0$ in the height formula, we expect to obtain two time values: the time at which the flare is shot into the air, and the time at which it returns to the ground. Let's see:

$$\begin{aligned}y &= 100t - 10t^2 \\0 &= 100t - 10t^2 \\0 &= 10(10t - t^2) \\0 &= 10t(10 - t)\end{aligned}$$

The zeros of the quadratic function are $t = 0$ (as expected), and $t = 10$. The latter zero can be obtained by setting the second factor, $(10 - t)$ equal to 0 and solving for t .

Thus, the flare spends 10 s in the air.

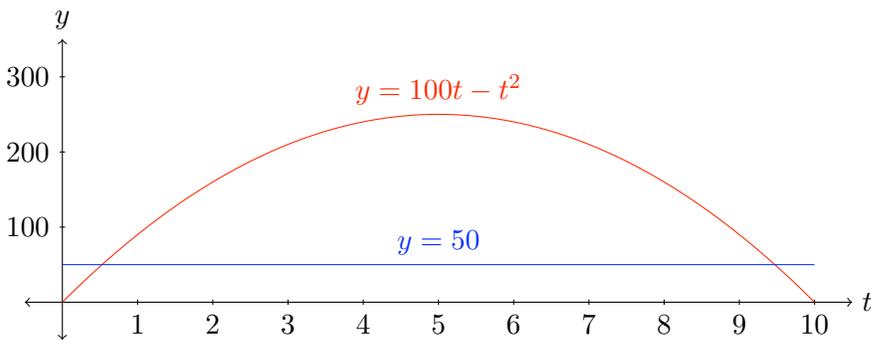
(b) This question is just like Part (a), except now we wish to know the times at which the height is 50 m. Subtracting the two times will tell us how long the flare was above the 50 m level. To determine these times, we can solve the equation

$$50 = 100t - 10t^2$$

which amounts to solving

$$0 = -50 + 100t - 10t^2$$

We'll learn how to solve such equations in subsequent modules. In the meantime, you could use software or a graphing calculator to estimate the solutions to the equation. You might like to approximate the values by reading from the graph in Figure . The calculated values are $t = 5 - 2\sqrt{5} \approx 0.528$ s and $t = 5 + 2\sqrt{5} \approx 9.472$ s. The difference between the two times, which is the amount of time that the flare spends above 50 m, is $4\sqrt{5} \approx 8.944$ s.



DISCUSSION PROBLEM

The following problem is open in the sense that there may be no definitive solution. Unlike typical textbook exercises, real-life problems rarely have cut-and-dried solutions. Discuss this problem with classmates or friends, then do your best to come up with a reasonable solution, and be prepared to identify and defend the assumptions you make.

Path of a projectile

Are the paths of soccer balls, baseballs, cannon balls, and other projectiles really parabolic?

WWW

- What we did: We learned how to determine the zeros of simple quadratic functions.
- Why we did it: Since quadratic functions describe simple motions involving projectiles, their zeros represent useful physical quantities.
- What's next: Determining the zeros of more complicated quadratic functions using factoring and the quadratic formula.

CHALLENGE PROBLEM

Imagine you toss a ball straight upwards so that it reaches a maximum height of 20 m. Assuming that there is no air resistance, determine how long it will take the ball to reach your hands again if you perform the experiment

- (a) on earth, where the acceleration due to gravity is approximately 9.8 m/s^2 .
- (b) on the moon, where the acceleration due to gravity is approximately 1.6 m/s^2 .